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1 Dynamical systems

We will consider both continuous and discrete dynamical systems.

1.1 Continuous dynamical systems

Continuous dynamical systems are defined by a system of autonomous (time–independent) ordinary differential equations (ODE)

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2, \ldots, x_n), \\
\dot{x}_2 &= f_2(x_1, x_2, \ldots, x_n), \\
&\vdots \\
\dot{x}_n &= f_n(x_1, x_2, \ldots, x_n),
\end{align*}
\]

or, in short,

\[\dot{x} = f(x), \quad \text{for} \quad x \in \mathbb{R}^n, \quad f : \mathbb{R}^n \to \mathbb{R}^n.\]

Assuming that this system of EDO can be integrated for all time, we can define the flow,

\[\phi_t(x), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n,\]

that satisfies

\[
\begin{align*}
\frac{d}{dt} \phi_t(x) &= f(\phi_t(x)), \\
\phi_0(x) &= x,
\end{align*}
\]

Given an initial condition \(x_0\), we can consider the corresponding orbit,

\[\{\phi_t(x_0)\}_{t \in \mathbb{R}}.\]

A fixed point of a continuous dynamical system (we will often say “of a flow”) is a point whose orbit is itself, that is,

\[\phi_t(x) = x, \quad \forall t \in \mathbb{R},\]

this can only happen if \(f(x) = 0\).

An orbit \(\{\phi_t(x)\}_{t \in \mathbb{R}}\) is said to be periodic if there is \(T > 0\) such that

\[\phi_T(x) = x, \quad \phi_t(x) \neq x, \quad \text{for} \quad 0 < t < T,\]

then \(T\) is said to be its period.

A set of initial conditions \(A \subset \mathbb{R}^n\) is said to be invariant if

\[\phi_t(x) \in A \quad \forall t \in \mathbb{R}, \quad \forall x \in A.\]

A trivial example is an orbit (in particular, a fixed point or a periodic orbit).

Consider a non–autonomous dynamical system,

\[\dot{x} = g(t, x), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n\]
with \( x = (x_1, \ldots, x_n)^\top \). In order to consider the dynamical system associated to a non-autonomous system of ODE, we first have to autonomize it. This can be done by taking \( y = (t, x)^\top = (y_0, y_1, \ldots, y_n)^\top \), and

\[
\begin{align*}
\dot{y}_0 &= 0, \\
\dot{y}_1 &= g_1(y_0, y_1, \ldots, y_n), \\
\vdots & \\
\dot{y}_n &= g_n(y_0, y_1, \ldots, y_n).
\end{align*}
\]

so we can consider the corresponding dynamical system. Note that such a dynamical system has never fixed points.

**Example 1.1** The RTBP in Hamiltonian form,

\[
\begin{align*}
\dot{x} &= \dot{p} + y, & \dot{p} &= p - \frac{1 - \mu}{r_1^2}(x - \mu) - \frac{\mu}{r_2^2}(x - \mu + 1), \\
\dot{y} &= - \dot{x}, & \dot{p}_y &= -p - \left(\frac{1 - \mu}{r_1^2} + \frac{\mu}{r_2^2}\right)y, \\
\dot{z} &= p_z, & \dot{p}_z &= -\left(\frac{1 - \mu}{r_1^2} + \frac{\mu}{r_2^2}\right)z.
\end{align*}
\]

whose Hamiltonian is

\[
H(x, y, z, p_x, p_y, p_z) = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) - xp_y + yp_x - \frac{1 - \mu}{r_1} + \frac{\mu}{r_2},
\]

being

\[
\begin{align*}
r_1 &= \sqrt{(x - \mu)^2 + y^2 + z^2}, \\
r_2 &= \sqrt{(x - \mu + 1)^2 + y^2 + z^2},
\end{align*}
\]

where \( \mu = m_2/m_1 \) is the mass parameter, if \( m_1 > m_2 \) are the masses of the primaries.

In short, the RTBP can be denoted as \( \dot{x} = f(x) \), where

\[
x = (x, y, z, p_x, p_y, p_z)^\top, \quad f(x) = (f_1(x), f_2(x), \ldots, f_n(x)),
\]

being

\[
f_1(x) = p_x + y, \quad \ldots, \quad f_6(x) = -\left(\frac{1 - \mu}{r_1^2} + \frac{\mu}{r_2^2}\right)z.
\]
Exemple 1.2 The Bicircular Problem in Hamiltonian form.

\[
\begin{align*}
\dot{x} &= p_x + y, \quad \dot{y} = p_y - x, \quad \dot{z} = p_z, \\
\dot{p}_x &= p_y - \frac{1 - \mu}{r_1^3} (x - \mu) - \frac{\mu}{r_2^3} (x - \mu + 1) - \frac{\eta}{r_3^3} (x - a \cos(\theta_0 + \omega t)) \\
&\quad - \frac{\eta}{a^2} \cos(\theta_0 + \omega t), \\
\dot{p}_y &= -p_x - \left( \frac{1 - \mu}{r_1^3} + \frac{\mu}{r_2^3} \right) y - \frac{\eta}{r_3^3} (y + a \sin(\theta_0 + \omega t)) + \frac{\eta}{a^2} \sin(\theta_0 + \omega t), \\
\dot{p}_z &= -\left( \frac{1 - \mu}{r_1^3} + \frac{\mu}{r_2^3} + \frac{\eta}{r_3^3} \right) z.
\end{align*}
\]

being

\[
\begin{align*}
r_1 &= \sqrt{(x - \mu)^2 + y^2 + z^2}, \\
r_2 &= \sqrt{(x - \mu + 1)^2 + y^2 + z^2}, \\
r_3 &= \sqrt{(x - a \cos(\theta_0 + \omega t))^2 + (y + a \sin(\theta_0 + \omega t))^2 + z^2} \\
\mu &= \frac{m_2}{m_1}, \quad \eta = \frac{m_3}{m_1 + m_2 + m_3},
\end{align*}
\]

with \( \omega \) the synodic frequency of \( m_3 \) and \( \theta_0 \) its phase for \( t = 0 \). The Hamiltonian is

\[
H(x, y, z, p_x, p_y, p_z) = \frac{1}{2} (p_x^2 + p_y^2 + p_z^2) - xp_y + yp_x - \frac{1 - \mu}{r_1^3} - \frac{\mu}{r_2^3} \\
- \frac{\eta}{a^2} (x \cos(\theta_0 + \omega t) - y \sin(\theta_0 + \omega t)).
\]

As we did with the RTBP, we can denote the differential equations of the BCP in short form as

\[
\dot{x} = f_\theta(t, x).
\]

The BCP can considered an autonomous system of ODE (and, therefore, a dynamical system) if we introduce a new variable \( \theta \) by

\[
\begin{align*}
\dot{\theta} &= \omega, \quad \dot{x} = p_x + y, \quad \dot{y} = p_y - x, \quad \dot{z} = p_z, \\
\dot{p}_x &= p_y - \frac{1 - \mu}{r_1^3} (x - \mu) - \frac{\mu}{r_2^3} (x - \mu + 1) - \frac{\eta}{r_3^3} (x - a \cos \theta) \\
&\quad - \frac{\eta}{a^2} \cos \theta, \\
\dot{p}_y &= -p_x - \left( \frac{1 - \mu}{r_1^3} + \frac{\mu}{r_2^3} \right) y - \frac{\eta}{r_3^3} (y + a \sin \theta) + \frac{\eta}{a^2} \sin \theta, \\
\dot{p}_z &= -\left( \frac{1 - \mu}{r_1^3} + \frac{\mu}{r_2^3} + \frac{\eta}{r_3^3} \right) z.
\end{align*}
\]
Since $\theta$ appears inside a $2\pi$–periodic function always, we can consider $\theta \in \mathbb{T} = \mathbb{R}/[0,2\pi]$, that is, given $\theta \in [0,2\pi]$ we consider $\theta - 2\pi, \theta + 2\pi, \theta - 4\pi, \theta + 4\pi, \ldots$ the same value. We say that $\theta$ is a phase.

1.2 Discrete dynamical systems.

Discrete dynamical systems are defined by diffeomorphisms (smooth 1–1 maps)

$$ F : \mathbb{R}^n \rightarrow \mathbb{R}^n $$

$$ x \mapsto F(x). $$

We denote by $F^{-1}$ the inverse map of $F$, and use superscript notation for the composition of maps:

$$ F^0(x) = x, $$

$$ F^1(x) = F(x), $$

$$ F^2(x) = F(F(x)), $$

$$ F^{-1}(x) = F^{-1}(F^{-1}(x)), $$

Some related concepts:

- Given an initial condition, its related orbit is the set $\{F^i(x)\}_{i \in \mathbb{Z}}$, that is,

$$ \{ \ldots, F^{-3}(x), F^{-2}(x), F^{-1}(x), F^0(x), F^1(x), F^2(x), F^3(x), \ldots \}. $$

- A fixed point is an initial condition such that its orbit is itself, that is $F(x) = x$.

- An $n$–periodic point is an initial condition $x$ such that $F^n(x) = x$ and $F^i(x) \neq x$ for $i = 1, \ldots, n - 1$. A fixed point is an 1–periodic point.

- A set of initial conditions $A \in \mathbb{R}^n$ is said to be an invariant set if $F^n(x) \in A$ for any $n \in \mathbb{Z}$ and $x \in A$. 

6
**Exemple 1.3** The standard map with parameter \( a = -0.7 \). It is defined by

\[
f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + a \sin(x + y) \\ x + y \end{pmatrix},
\]

for \( x, y \in \mathbb{T} = \mathbb{R}/[0, 2\pi] \).

\[
\begin{pmatrix}
\hline
-3 & -2 & -1 & 0 & 1 & 2 & 3 \\
\hline
-3 & -2 & -1 & 0 & 1 & 2 & 3
\end{pmatrix}
\]

\[
\begin{pmatrix}
\hline
\theta & x & y & z & p_x, p_y, p_z \\
\hline
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{d}{dt} \\
\hline
\theta & x & y & z & p_x & p_y & p_z \\
\hline
\end{pmatrix}
\]

being

\[
\begin{align*}
g_0(\theta, \ldots, p_z) &= \omega, \\
g_1(\theta, \ldots, p_z) &= p_x + y, \\
g_2(\theta, \ldots, p_z) &= p_y - x, \\
& \vdots \\
\end{align*}
\]

Denote it as \( \dot{y} = g(y) \) and consider the corresponding flow \( \Phi_t(y) \), defined by

\[
\frac{d}{dt} \Phi_t(y) = F(\Phi_t(y)), \\
\Phi_0(y) = y, \quad \forall y \in \mathbb{R}^7.
\]
In fact,
\[ \Phi_t(y) = \left( \begin{array}{c} \theta + \omega t \\ \phi_t^\theta(x) \end{array} \right), \]
where \( \phi_t^\theta(x) \) is the flow from time 0 to time \( t \) of the BCP with starting phase \( \theta \):
\[
\frac{d}{dt} \phi_t^\theta(x) = f_\theta(t, \phi_t^\theta(x)), \\
\phi_0^\theta(x) = x, \quad \forall x \in \mathbb{R}^6.
\]
We can consider the time–2\( \pi/\omega \) flow of \( \dot{y} = g(y), \ G(y) = \Phi_{2\pi/\omega}(y) \), that is,
\[
G \left( \begin{array}{c} \theta \\ x \end{array} \right) = \left( \begin{array}{c} \theta + \frac{2\pi}{\omega} \\ \phi_{2\pi/\omega}^\theta(x) \end{array} \right) = \left( \begin{array}{c} \theta \\ \phi_{2\pi/\omega}^\theta(x) \end{array} \right).
\]
Since the \( \theta \) coordinate is invariant, we can skip it and define
\[
F_\theta(x) := \phi_{2\pi/\omega}^\theta(x).
\]
Note that a fixed point of \( F_\theta \) is a periodic orbit of the BCP with initial phase \( \theta \).

### 1.3 Orbit generation in a dynamical system

For a discrete system, we just need to write a routine that evaluates the corresponding map and iterate it.

For a continuous system, we need to use numerical methods for the integration of ODEs. One of the most popular methods in the context of astrodynamics is the Runge–Kutta–Fehlberg (RKF) method of orders 7 and 8, which was developed by Fehlberg in 1968 under contract by NASA for the Apollo missions.

We will not enter in the details of the implementation of a RKF routine, but just comment its use as a black box. Consider a system of \( n \) possibly non–autonomous ODE,
\[
\dot{x} = f(t, x),
\]
and denote by \( \phi_{t_0}^t(x) \) its flow from time \( t_0 \) to time \( t \), such that
\[
\frac{d}{dt} \phi_{t_0}^t(x) = f(t, \phi_{t_0}^t(x)), \\
\phi_{t_0}^{t_0}(x) = x, \quad \forall x \in \mathbb{R}^n.
\]
Given \( t_0 \in \mathbb{R}, \ x_0 \in \mathbb{R}^n, \ h_0 \in \mathbb{R} \) (small), and a tolerance \( \delta \), the RKF routine returns \( t_1, x_1, h_1 \) verifying
(a) \( |x_1 - \phi_{t_0}^{t_1}(x_0)| < \delta \),
(b) \( t_1 \) is as close to \( t_0 + h_0 \) as possible,
(c) $h_1$ is a recommended steplength for the next call.

In the algorithmic descriptions that will follow, we will denote a call to the RKF routine as

$$(t_1, x_1, h_1) = \text{RKFstep}(t_0, x_0, h_0, f, \text{tol}).$$

In order to implement $\phi_{t_0}^T(x_0)$ for arbitrary $T, T_0, x_0$, we need to write a routine that calls the RKF routine as many times as needed. In some algorithmical descriptions that will follow, we will denote a call to such a routine as

$$(t_1, x_1, h_1) = \text{RKFflow}(t_0, t, x_0, h_0, f, \text{tol}).$$

In what follows, we will often need the derivative of the flow with respect to initial conditions, that we will denote as $D\phi_t(x)$. It can be found by solving the variational equations:

$$\begin{align*}
\dot{x} &= f(t, x), \\
\dot{A} &= \frac{\partial f}{\partial x}(t, x)A,
\end{align*}$$

where $x$ is an $n$–dimensional vector and $A$ is a $n \times n$ matrix. If $x(t)$ and $A(t)$ are solutions of (1) with $x(t_0) = x_0$ and $A(t_0) = I_n$ (the $n \times n$ identity matrix), then

$$D\phi_{t_0}^t(x_0) = A(t).$$

The variational equations can be written as a system of $n + n^2$ ODE as

$$\begin{align*}
\dot{x}_1 &= f_1(t, x_1, \ldots, x_n), \\
\dot{a}_{1,1} &= \sum_{k=1}^n \frac{\partial f_1}{\partial x_k}(t, x)a_{k,1}, \ldots, \\
\dot{a}_{1,n} &= \sum_{k=1}^n \frac{\partial f_1}{\partial x_k}(t, x)a_{k,n}, \\
\vdots &= \vdots \\
\dot{x}_n &= f_n(t, x_1, \ldots, x_n), \\
\dot{a}_{n,1} &= \sum_{k=1}^n \frac{\partial f_n}{\partial x_k}(t, x)a_{k,1}, \ldots, \\
\dot{a}_{n,n} &= \sum_{k=1}^n \frac{\partial f_n}{\partial x_k}(t, x)a_{k,n}.
\end{align*}$$

### 1.4 Poincaré maps

Consider a continuous dynamical system given by

$$\dot{x} = f(x).$$

A first method to obtain a discrete dynamical system from a continuous one is by considering its time–$T$ flow. We have already done that in example 1.2.

Another method, which is convenient in many situations, is through a Poincaré section. Let $\Sigma$ be a hypersurface of $\mathbb{R}^n$, and assume it is transversal to the vectorfield, that is, the vectorfield is not tangent $\Sigma$ in any point of $\Sigma$,

$$\forall x \in \Sigma \quad f(x) \notin T_x(\Sigma).$$
Let $x_0$ be such that $\phi_{T_0} \in \Sigma$ for some $T_0 > 0$, and assume that $T_0$ is minimum with this property. Under suitable hypothesis, there exists a neighborhood $U \ni x_0$ and a map $\tau : U \to \mathbb{R}^n$, known as time-return map, such that
\[ \phi_{\tau(x)}(x) \in \Sigma \quad \forall x \in U. \]

The map
\[ P(x) = \phi_{\tau(x)}(x) \]
is called Poincaré map or first-return map corresponding to $\Sigma$. The restriction of $P$ to $V := \Sigma \cap U$ defines a discrete dynamical system.

Note that a fixed point of $P$ is an initial condition for a periodic orbit of the starting continuous dynamical system.

1.4.1 Numerical computation of a Poincaré map

Remember that the Poincaré map is defined by
\[ P(x) = \phi_{\tau(x)}(x). \]
Assume that $\Sigma$ is given by \( \{ g(x) = 0 \} \) and is to be traversed from \( \{ g(x) < 0 \} \) to \( \{ g(x) > 0 \} \). Then, the evaluation of the Poincaré map can be done as in the following algorithm:

**input:** $x, g, f, \text{tol}$
**do:**
- $t:=0, y:=x, h:=\text{tol}$
- while $(g(y) \geq 0)$
  - $(t,y,h):=\text{RKFstep}(t,y,h,f,\text{tol})$
- while $(g(y) < 0)$
  - $(t,y,h):=\text{RKFstep}(t,y,h,f,\text{tol})$
- while $(|g(y)| < \text{tol})$
  - $\delta:=-\frac{g(y)}{Dg(y)f(y)}$
  - $(t,y,h):=\text{RKFflow}(t,t+\delta,y,h,f,\text{tol})$
**output:** $t, y$

Each iteration of the last loop corresponds to performing a Newton iteration to find a zero of the function $\delta \mapsto \phi_{\delta}(y)$ starting from $\delta = 0$. At the end of the algorithm, $y = P(x)$ and $t = \tau(x)$.

In what follows, we will also need the differential of the Poincaré map. It can be computed as
\begin{align*}
DP(x) &= \frac{d}{dx} \phi_{\tau(x)}(x) = \frac{d}{d\tau} \phi_{\tau(x)}(x) + D\phi_{\tau(x)}(x) \\
&= f(P(x)) D\tau(x) + D\phi_{\tau(x)}(x),
\end{align*}
(2)
so we need the differential of the time–return map. We can obtain it by differentiating the condition of a Poincaré section:

\[
0 \equiv g(P(x)) \\
\implies 0 = Dg(P(x))DP(x) \\
= Dg(P(x))\left( f(P(x))D\tau(x) + D\phi_{\tau(x)}(x) \right) \\
= \left( \begin{array}{c} Dg(P(x)) \cr f(P(x)) \end{array} \right) \left( \begin{array}{c} D\tau(x) \\ n \times 1 \end{array} \right) + \left( \begin{array}{c} Dg(P(x)) \cr D\phi_{\tau(x)}(x) \end{array} \right) \left( \begin{array}{c} x \cr n \times 1 \end{array} \right) \\
\implies D\tau(x) = -\frac{Dg(P(x))D\phi_{\tau(x)}(x)}{Dg(P(x))f(P(x))},
\]

and, by substituting in (3),

\[
DP(x) = -f(P(x))\left( \begin{array}{c} Dg(P(x)) \cr D\phi_{\tau(x)}(x) \end{array} \right) + D\phi_{\tau(x)}(x)
\]

If, in the previous algorithm for the evaluation of a Poincaré map, we integrate also the variational equations, we get both \( \tau(x) \) and \( D\phi_{\tau(x)}(x) \).

2 Computation of objects

Gallery of objects:

- Fixed points (RTBP: \( L_1, L_2, L_3, L_4, L_5 \)).
- Periodic orbits (RTBP: Lyapunov planar, vertical, halo).
- Invariant tori (RTBP: Lissajous, quasi-halo).
- Invariant manifolds (in Josep’s session).
- Homoclinic and heteroclinic phenomena.

2.1 Computation of fixed points

For the computation of a fixed point of a flow \( \dot{x} = f(x) \), we look for \( p \) such that \( f(p) = 0 \). For the computation of a fixed point of a map \( x \mapsto F(x) \), we look for \( p \) such that \( 0 = G(p) := F(p) - p \).

In any case, we can use Newton’s method in several variables to look for a zero of \( G \):
\begin{verbatim}
input: \( p_0, G, \) tol, maxit
do:  \( p := p_0 \)
   for it from 1 to maxit do
      if (\(|G(p)| > \text{tol}\)) return \( p \)
      solve \( D G(p) \Delta p = G(p) \) for \( \Delta p \)
      \( p := p - \Delta p \)
   error (maxit exceeded)
output: \( p \) (if OK)
\end{verbatim}

In the RTBP, it can be analytically seen that the distance from \( L_j, j = 1, 2, 3, \) to the closest primary, which we call \( \gamma_j, \) is given by the only positive root of the corresponding Euler’s quintic equation:
\[
\begin{align*}
\gamma_j^5 &\mp (3 - \mu) \gamma_j^4 + (3 - 2\mu) \gamma_j^3 - \mu \gamma_j^2 \pm 2\mu \gamma_j - \mu \quad = \quad 0, \quad j = 1, 2, \\
\gamma_j^5 + (2 + \mu) \gamma_j^4 + (1 + 2\mu) \gamma_j^3 - (1 - \mu) \gamma_j^2 - 2(1 - \mu) \gamma_j - (1 - \mu) \quad = \quad 0, \quad j = 3.
\end{align*}
\]
Therefore, in this case it is enough to use Newton’s method in one dimension. Good guesses are \((\mu/3)^{1/3}\) for \( L_{1,2} \) and \( 1 - (7/12)\mu \) for \( L_3. \)

2.2 Linear behaviour around a fixed point

By studying the linear behavior around a fixed point we can learn which kind of objects originate from them (such as invariant manifolds, periodic orbits or invariant curves), and obtain good guesses for its computation.

2.2.1 Flows

Consider a flow
\[
\dot{x} = f(x),
\]
with a fixed point \( p, \)
\[
f(p) = 0,
\]
The Taylor expansion of \( f \) around \( p \) up to order one is
\[
f(x) = f(p) + Df(p)(x - p) + O(\|x - p\|^2),
\]
so that the linearized flow around \( p \) is
\[
\dot{x} = A(x - p).
\]
The eigenvalues of \( A \) are known as the **exponents** of the fixed point \( p.\)

Assume that \( \lambda \neq 0 \) is an eigenvalue of \( A, \) and \( v \) is a corresponding eigenvector.

- If \( \lambda \in \mathbb{R}, \) consider \( \varphi(t) = p + e^{\lambda t}v. \) Then:
- $\varphi(t)$ satisfies the system of ODE of the linearized flow,

$$\varphi'(t) = \lambda e^\lambda v = e^\lambda(\lambda v) = e^\lambda(Av) = A(e^\lambda v) = A(\varphi(t) - p).$$

- If $\lambda > 0$, $\varphi(t) \xrightarrow{t \to \infty} p$, so that it gives a stable manifold of the linearized flow.
- If $\lambda < 0$, $\varphi(t) \xrightarrow{t \to -\infty} p$, so that it gives an unstable manifold of the linearized flow.

Under suitable hypothesis of the remaining eigenvalues, the existence of a stable or unstable manifold of the full (nonlinear) dynamical system is ensured by the stable manifold theorem for flows.

- If $\lambda = i\omega$, for $\omega \in \mathbb{R}$, let $v_1 + iv_2$ be a corresponding eigenvector, with $v_1, v_2 \in \mathbb{R}^n$. Then

$$Av_1 + iAv_2 = A(v_1 + iv_2) = i\omega(v_1 + iv_2) = -\omega v_2 + i\omega v_1.$$

Therefore, if we define

$$\varphi_\gamma(t) = p + \gamma((\cos \omega t)v_1 + (\sin \omega t)v_2),$$

we have

$$\varphi'(t) = \gamma(\omega(\cos \omega t)v_1 - \omega(\sin \omega t)v_2) = \gamma((\cos \omega t)Av_2 + (\sin \omega t)Av_1) = A(\gamma((\sin \omega t)v_1 + (\cos \omega t)v_2)) = A(\varphi(t) - p),$$

so that $\varphi(t)$ satisfies the linearized system of ODE. It thus gives a one-parametric families of periodic orbits of period $2\pi/\omega$ of the linear approximation.

Under suitable non-resonance conditions with respect to the remaining eigenvalues, the existence of a family of periodic orbits for the full nonlinear system, with limiting period $2\pi/\omega$, is ensured by Liapunov’s center theorem.

In a Hamiltonian system, if $\lambda$ is an eigenvalue of $Df(p)$, so is $-\lambda$. Therefore, to each stable (unstable) manifold there is a corresponding unstable (stable) manifold.

- The case $\lambda = a + i\omega$ for $a, \omega \in \mathbb{R}, a, \omega \neq 0$ is impossible in a Hamiltonian system, and we will not consider it here.

The existence of the nonlinear equivalents of the manifolds found for the linear approximation is given by the following

**Theorem 2.1 (Center manifold theorem for flows)** Let $\dot{x} = f(x)$ be a continuous dynamical system, $f(p) = 0$. Divide the eigenvalues of $f(p)$ in three parts,

$$\text{Spec } Df(p) = \sigma_s \cup \sigma_c \cup \sigma_u,$$

such that
• $\text{Re} \lambda < 0$ for $\lambda \in \sigma_s$,
• $\text{Re} \lambda = 0$ for $\lambda \in \sigma_c$,
• $\text{Re} \lambda > 0$ for $\lambda \in \sigma_u$.

Denote by $\langle \sigma_s \rangle$ the linear subspace spanned by the eigenvectors of eigenvalues in $\sigma_s$, and analogously for $\langle \sigma_c \rangle$ and $\langle \sigma_u \rangle$.

Then, in a neighborhood of there exist three manifolds

• $W^s$, tangent to $\langle \sigma_s \rangle$ (therefore, of the same dimension) at $p$, known as the local stable manifold of $p$,
• $W^c$, tangent to $\langle \sigma_c \rangle$ at $p$, known as the local center manifold of $p$,
• $W^u$, tangent to $\langle \sigma_u \rangle$ at $p$, known as the local unstable manifold of $p$,

The stable and unstable manifolds satisfy

• $\forall x \in W^s, \phi_t(x) \to +\infty \to p$,
• $\forall x \in W^u, \phi_t(x) \to -\infty \to p$.

In the case of purely imaginary eigenvalues of a Hamiltonian, the nonlinear equivalent of the family of periodic orbits of the linear approximation is given by the following

**Theorem 2.2 (Liapunov’s Center Theorem)** Let $\dot{x} = f(x)$ be an autonomous Hamiltonian dynamical system, $p$ a fixed point of it and assume that:

• The exponents of $p$ are $\omega i, -\omega i, \lambda_2, \bar{\lambda}_2, \ldots, \lambda_n, \bar{\lambda}_n$, with $\omega > 0$.
• $\lambda_j/(i\omega)$ is never an integer for $j = 2 \div n$.

Then

• There exists a one-parametric family of p.o. emanating from $p$.
• The period of the p.o. of the family tend to $2\pi/\omega$ when approaching $p$.
• Their nontrivial multipliers tend to $e^{2\pi \lambda_j/\omega}$, $j = 2 \div n$, when approaching $p$. 

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2.2.2 Maps

Consider a discrete dynamical system given by

\[ x \mapsto F(x), \]

with a fixed point \( p \),

\[ F(p) = p. \]

The taylor expansion of \( F \) around \( p \) up to order one is

\[ F(x) = F(p) + \frac{DF(p)(x - p)}{p} + O(\|x - p\|^2), \]

so that the linearized dynamical system around \( p \) is

\[ x \mapsto L_F(x) := p + A(x - p). \]

The eigenvalues of \( A \) are also called the multipliers of the fixed point \( p \).

Let \( \lambda \) be a nonzero eigenvalue of \( A \), and \( v \) a corresponding eigenvector.

- If \( \lambda \in \mathbb{R}, |\lambda| \neq 1 \), consider \( \varphi(\xi) = p + \xi v \). Then,

\[ L_F(\varphi(\xi)) = p + A(\varphi(\xi) - p) = p + A\xi v = p + \xi Av = p + (\xi \lambda)v = \varphi(\lambda \xi), \]

so that \( \{\varphi(\xi)\}_{\xi \in \mathbb{R}} \) is an invariant set of the linear approximation. Moreover,

- If \( |\lambda| < 1 \),

\[ L^n_F(\varphi(\xi)) = \varphi(\lambda^n \xi) \xrightarrow{n \to \infty} \varphi(0) = p, \]

and therefore \( \varphi \) parametrizes a stable manifold of \( p \) for the linearized map.

- If \( |\lambda| > 1 \),

\[ L^n_F(\varphi(\xi)) = \varphi(\lambda^n \xi) \xrightarrow{n \to -\infty} \varphi(0) = p, \]

and therefore \( \varphi \) parametrizes an unstable manifold of \( p \) for the linear map.

Under suitable hypothesis on the remaining eigenvalues, the existence of stable and unstable manifolds for the full nonlinear map is ensured by the stable manifold theorem for maps.

- If \( \lambda \in \mathbb{C}, |\lambda| = 1 \), assume \( \lambda = \cos \rho + i \sin \rho \) and let \( v_1 + iv_2 \) an associated eigenvector. Since

\[ Av_1 + iAv_2 = A(v_1 + iv_2) = (\cos \rho + i \sin \rho)(v_1 + iv_2) = (\cos \rho)v_1 + (\sin \rho)v_2 + i((\sin \rho)v_1 + (\cos \rho)v_2) \]

we have

\[ A[v_1, v_2] = [v_1, v_2] \begin{pmatrix} \cos \rho & \sin \rho \\ -\sin \rho & \cos \rho \end{pmatrix} R_{\rho}, \]
where we denote by \([v_1, v_2]\) the matrix with columns \(v_1, v_2\) and by \(R_\rho\) a clockwise rotation of angle \(\rho\). We define
\[
\varphi(\xi) = p + \gamma((\cos \xi)v_1 - (\sin \xi)v_2) = p + [v_1, v_2]R_\xi \begin{pmatrix} \gamma \\ 0 \end{pmatrix}.
\]
If we apply the linearized map to \(\varphi(\xi)\), we get
\[
\varphi(\xi) \mapsto p + A(\varphi(\xi) - p) = p + A[v_1, v_2]R_\xi \begin{pmatrix} \gamma \\ 0 \end{pmatrix} = p + [v_1, v_2]R_\rho R_\xi \begin{pmatrix} \gamma \\ 0 \end{pmatrix} = \varphi(\xi + \rho),
\]
so \(\varphi\) parametrizes an invariant closed curve of the linear flow.

We therefore have a one–parametric family (with parameter \(\gamma\)) of curves, with rotation number \(\rho\), of invariant by the linearized map. Under generic non–degeneracy conditions, this gives a one–parametric family of invariant curves for the full (non-linear) map, with varying rotation number which converges to \(\rho\) as we approach the fixed point.

### 2.3 Computation of periodic orbits (of flows)

They are found as a fixed point of a suitable discrete dynamical system.

#### 2.3.1 Non–autonomous case

Consider a non–autonomous \(T\)–periodic system of \(n\) ODE (e.g. the RTBP), that is
\[
\dot{x} = f(t, x),
\]
with
\[
f(t + T, x) = f(t, x), \quad \forall t \in \mathbb{R}, \quad \forall x \in \mathbb{R}^n.
\]
Consider the map
\[
F(x) = \phi_T^0(x).
\]
Then, in order to find a \(T\)–periodic orbit of \(\dot{x} = f(t, x)\), we look for a fixed point of \(F\) by using Newton’s method to find a zero of \(G(x) := F(x) - x = \phi_T^0(x) - x\). The derivative of \(G\) at the approximation \(p\) of the solution, corresponding to an iterate of Newton’s method, can be computed by integrating the variational equations,
\[
\dot{x} = f(t, x),
\]
\[
\dot{A} = Df(t, x)A,
\]
\[
x(0) = p, \quad A(0) = I,
\]
then,
\[
D\phi_T^0(p) = A(T),
\]
and therefore,
\[
DG(p) = A(T) - I.
\]
2.3.2 Autonomous case

Assume we have an autonomous system of ODE

\[ \dot{x} = f(x), \]

and consider its flow \( \phi_t(x) \), defined by

\[
\frac{d}{dt} \phi_t(x) = f(\phi_t(x)),
\]

\[ \phi_0(x) = x. \]

The previous approach does not work because the whole periodic orbit is a solution of \( G(p) = 0 \), so \( DG(p) = 0 \) is singular (its kernel gives direction of the tangent line to the periodic orbit through \( p \)) and this gives problems in solving the corresponding linear system of Newton's method.

The solution to this problem is to consider a Poincaré section. Let \( \Sigma \) be an hypersurface or \( \mathbb{R}^n \) transversal to the flow, and let \( x_0 \) an initial condition of the p.o. that lies in the section. Let \( \tau(x) \) be the time–return map, so that the Poincaré map is written as

\[ P(x) = \phi_{\tau(x)}(x). \]

Then \( x_0 \) is the only point of the p.o. that is also a fixed point of the Poincaré map.

Then, a periodic orbit of the original continuous system is found as a fixed point of the Poincaré map by looking for a zero of the function \( G(x) := P(x) - x \) by Newton's method. Remember that, if the hypersurface of section is given by \( \{g(x) = 0\} \),

\[ D\tau(x) = -\frac{Dg(P(x))D\phi_{\tau(x)}(x)}{Dg(P(x))f(P(x))}, \]

and therefore

\[ DP(x) = -f(P(x))\frac{Dg(P(x))D\phi_{\tau(x)}(x)}{Dg(P(x))f(P(x))} + D\phi_{\tau(x)}(x). \]

This would work in order to find an isolated o.p. But, in autonomous Hamiltonian systems (like the RTBP or Hill’s problem), periodic orbits are not isolated but embedded in families, which give a curve of fixed points in the Poincaré section. So, again, the differential of the restriction of \( G \) to the hypersurface is singular at any of these points (its kernel gives the direction of the tangent line to the point), and we have trouble in solving the linear system in Newton’s method.

The solution to this problem is to add an additional constraint in order to have an unique o.p. as solution. It can be, either

- To prescribe a certain period.
- To prescribe an energy level.
**Practical implementation**  Consider the RTBP (or, equivalently, any autonomous Hamiltonian system). We can write a routine that evaluates a system of equations implementing the two constraints mentioned, together with its differential:

\[
\begin{align*}
H(x) - h &= 0 \\
\tau(x) - T &= 0 \\
\phi_t(x)(x) - x &= 0
\end{align*}
\]

with unknowns \((h, T, x) = (h, T, x, y, z, p_x, p_y, p_z)\). This system, as is, does not need to be compatible. In fact, generically the period (locally) determines a unique periodic orbit in the family, as well as the energy does. What we can do is to eliminate equations and unknowns in the previous system in order to obtain equations for a particular approach. Here,

- by “eliminating” an equation, we mean exactly this, and
- by “eliminating” an unknown, we mean to keep it constant in Newton’s method, as if it were a parameter.

In this way, for instance:

- To compute a p.o. of a given energy level, we eliminate equation 2 and unknowns \(h, T\).
- To compute a p.o. of a given period, we eliminate equation 1 and unknowns \(h, T\).
- To compute a p.o. of a given energy level an a prescribed value of a coordinate, we eliminate the second equation and the unknowns \(h\) and the prescribed coordinate.

And all this can be done by the same routine.

Note that, with any of the previous approaches, we end up with a linear system that we know has an unique solution, but is not square (has one more equation than unknowns). So we cannot use a standard \(LU\) or \(QR\) decomposition, for instance. This is solved by using a routine to find the minimum–norm least–squares solution of an arbitrary linear system.

### 2.4 Multiple shooting

In the neighborhood of the collinear points of the RTBP, the maximum eigenvalue of the monodromy matrices of p.o. can be as large as 2000 or even larger. This means that any error in the initial condition is amplified by this factor. Even with exact data, the local truncation error of the first RKF step is amplified by this factor.\(^1\) In particular, we cannot expect an error smaller than \(10^{-11}\), assuming the tolerance of the RKF routine is set to \(10^{-14}\).

\(^1\)Even the first floating point operation, which can have a relative error up to the epsilon of the machine, is amplified by this factor.
We can reduce these amplification factors by making use of multiple shooting. In the context of p.o., this means that, instead of looking for \( h, T, x \), we look for \( h, T, x_0, \ldots, x_{m-1} \), for \( m > 1 \), satisfying

\[
\begin{align*}
H(x_0) - h &= 0 \\
T(x_{m-1}) - \frac{T}{m} &= 0 \\
\phi_{T/m}(x_i) - x_{i+1} &= 0, \quad i = 0 \div m - 2 \\
\phi_T(x_{m-1})(x_{m-1}) - x_0 &= 0
\end{align*}
\]

Whis this approach, one obtains amplification factors that are, typically, the \( m \)-th root of the starting ones.

### 2.5 Linear behaviour around a p.o. of an autonomous system

An initial condition \( x_0 \) of an o.p. is also a fixed point of \( \phi_T \). Although this is not useful to numerically find the p.o., as we have seen, it is useful to study the linear behaviour around it.

Consider the RTBP (or, equivalently, any autonomous Hamiltonian system), and let \( x_0 \) be an initial condition of a \( T \)-periodic orbit. Then, its monodromy matrix,

\[
M := D\phi_T(x_0)
\]

has 1 as double eigenvalue.

Moreover, \( M := D\phi_T(x_0) \) is a symplectic matrix, which implies that, if \( \lambda \) is an eigenvalue of \( M \), then \( 1/\lambda \) is also eigenvalue. Therefore, the eigenvalues of \( M \) are

\[
\{1, 1, \lambda_1, \lambda_1^{-1}, \lambda_2, \lambda_2^{-1}\},
\]

and we will assume that \( |\lambda_i| \leq |\lambda_i^{-1}| \).

The linear behaviour around a p.o. is better studied in terms of its stability parameters, \( s_1 \) and \( s_2 \), which are defined as

\[
s_1 = \lambda_1 + 1/\lambda_1, \quad s_2 = \lambda_2 + 1/\lambda_2.
\]

It is easy to check that

\[
\begin{align*}
s_i \in \mathbb{R}, \quad |s_i| > 2 &\iff \lambda_i \in \mathbb{R}\setminus\{-1, 1\}, \\
s_i \in \mathbb{R}, \quad |s_i| \leq 2 &\iff \lambda_i \in \mathbb{C}, \quad |\lambda_i| = 1, \\
s_i \in \mathbb{C}\setminus\mathbb{R} &\iff \lambda_i \in \mathbb{C}\setminus\mathbb{R}, \quad |\lambda_i| \neq 1.
\end{align*}
\]

Therefore:

- If \( s_i \in \mathbb{R}, |s_i| > 2 \) (hyperbolic case)
  - There is a stable manifold of the fixed point of \( \phi_T \), tangent to the \( \lambda_1 \)-eigendirection at \( x_0 \).
– There is an unstable manifold of the fixed point $\phi_T$, tangent to the $\lambda_2$–eigendirection at $x_0$.

In terms of the p.o.:

– There is a stable manifold of the p.o. whose section through the $\lambda_1, \lambda_2$–eigenplane is tangent to the $\lambda_1$–eigendirection.
– There is an unstable manifold of the p.o. whose section through the $\lambda_1, \lambda_2$–eigenplane is tangent to the $\lambda_2$–eigendirection.

• If $s_i \in \mathbb{R}$, $|s_i| \leq 2$ (*elliptic case*), let $\lambda_i = \cos \rho + i \sin \rho$. As we have seen, there is a continuous, one–parametric family of closed curves invariant by the linearization of $\phi_T$ around $x_0$ in the $\{x_0 + \alpha_1 \Re v_1 + \alpha_2 \Im v_2\}_{\alpha_1, \alpha_2 \in \mathbb{R}}$ plane, with rotation number $\rho$.

Under generic non–degeneracy conditions for $\rho$ (which, essentially, mean that the frequencies have to “move enough” along the family), there is a Cantorian family of invariant curves around $x_0$, with limiting rotation number $\rho$. When transported by the flow, these invariant curves generate two–dimensional invariant tori.

Rational values for $\rho$ give rise to bifurcations of the family of periodic orbits. The particular values $\rho = 0$ and $\rho = \pi$, which correspond to $s_i = 2$ and $s_i = -2$, respectively, are known as the *parabolic case*.

2.6 Numerical computation of 2D invariant tori

2.6.1 Looking for a parametrization of an invariant curve

We will develop the methodology for the RTBP (analogously, for any Hamiltonian autonomous system).

\[
\psi : \mathbb{R}^2 \rightarrow \mathbb{R}^6 \\
(\theta_1, \theta_2) \mapsto \psi(\theta_1, \theta_2),
\]

$\psi$ being a $2\pi$–periodic function in the $\theta_1, \theta_2$ variables. We could find such a $\psi$ by solving the functional equation

\[
\psi(\theta_1 + t\omega_1, \theta_2 + t\omega_2) = \phi_t(\psi(\theta_1, \theta_2)), \quad \forall t \in \mathbb{R}, \quad \forall \theta_1, \theta_2 \in [0, 2\pi],
\]

where $\omega_1, \omega_2$ are the frequencies of the torus.

In order to reduce the dimension of the problem, we observe that $\varphi(\xi) = \psi(\xi, 0)$ is a curve invariant by $\phi_{2\pi/\omega_2}$, and satisfies

\[
\varphi(\xi + \rho) = \phi_\delta(\varphi(\xi)),
\]

for $\rho = 2\pi\omega_1/\omega_2$ and $\delta = 2\pi/\omega_2$. 
Once we have $\varphi$, we can recover $\psi$ by

$$
\psi(\theta_1, \theta_2) = \phi_t\left(\varphi(\theta_1 - \frac{\theta_2}{2\pi})\right)
$$

Note that

$$
\varphi(\xi + \rho) = \phi_\delta(\varphi(\xi)),
$$

is a functional equation: we have “infinite equations” (one for each value of $\phi \in [0, 2\pi]$) and “infinite unknowns” (we cannot describe a generla function $\varphi$ by a finite number of parameters). We discretize function space by looking for $\varphi$ as a truncated Fourier series,

$$
\varphi(\xi) = A_0 + \sum_{k=1}^{N_f} \left( A_k \cos(k\xi) + B_k \sin(k\xi) \right).
$$

We will discretize parameter space by looking for $\varphi$ satisfying

$$
\varphi(\xi + \rho) - \phi_\delta(\varphi(\xi_i)), \quad i = 0 \div 2N_f,
$$

for $\xi_i = i2\pi/(1 + 2N_f)$.

### 2.6.2 Indeterminations

We have two indeterminations to cope with:

- Assuming there exists a parametrization of a 2D torus,

  $$
  \varphi(\theta_1 + t\omega_1, \theta_2 + t\omega_2) = \phi_t(\psi(\theta_1, \theta_2)),
  $$

  not only $\varphi(\xi) = \psi(\xi, 0)$ satisfies $\varphi(\xi + \rho) = \phi_\delta(\varphi(\xi))$, but any $\varphi(\xi) := \psi(\xi, \eta_0)$ for $\eta_0 \in [0, 2\pi]$ also does.

- If $\varphi(\xi)$ satisifes $\varphi(\xi + \rho) = \phi_\delta(\varphi(\xi))$, then, for any $\xi_0 \in \mathbb{R}$, $\varphi_{\xi_0}(\xi) = \varphi(\xi - \xi_0)$ also does.

  The first indetermination can be avoided by fixing a curve on the torus. This can be done by prescribing a value for a coordinate of $A_0$. It must be chosen by geometrical considerations.

  The second indetermination can be avoided by prescribing a coordinate of $A_1$ to be zero. Assume that $A_1 = (A_1^1, \ldots, A_1^6), \ B_1 = (B_1^1, \ldots, B_1^6)$. If $(A_1^k, B_1^k) \neq (0, 0)$, since

  $$
  A_1^k \cos(k(\xi - \xi_0)) + B_1^k \sin(k(\xi - \xi_0))
  = (A_1^k \cos k\xi_0 - B_1^k \sin k\xi_0) \cos k\xi + (A_1^k \sin k\xi_0 + B_1^k \cos k\xi_0) \sin k\xi
  =: \tilde{A}_1^k \cos k\xi + \tilde{B}_1^k \sin k\xi.
  $$

  we can always choose $\xi_0$ such that $\tilde{A}_1^k = 0.$

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2.6.3 The system of equations

Before the statement of the system of equations to be solved for the computation of (an invariant curve inside) an invariant torus, we make two more considerations:

- We want to be able to prescribe values for the energy, so we will add an equation for that.
- We want to overcome the effects of instability, so we will use multiple shooting.

We will, therefore, look for \( \varphi_0, \ldots, \varphi_{m-1} \) satisfying

\[
\begin{align*}
H(\varphi_0(0)) - h &= 0 \\
\varphi_{j+1}(\xi_i) - \phi_{\delta/m}(\varphi_j(\xi_i)) &= 0, \quad j = 0 \div m - 2, \quad i = 0, \ldots, 2N_f, \\
\varphi_0(\xi_i + \rho) - \phi_{\delta/m}(\varphi_{m-1}(\xi_i)) &= 0, \quad i = 0 \div 2N_f,
\end{align*}
\]

where

\[
\xi_i = i \frac{2\pi}{1 + 2N_f}, \quad i = 0 \div 2N_f,
\]

with unknowns

\[
h, \delta, \rho, A^0_0, A^0_1, B^0_1, \ldots, A^{m-1}_0, A^{m-1}_1, B^{m-1}_1, \ldots, A^{m-1}_{N_f}, B^{m-1}_{N_f}.
\]

with \( h, \delta \rho \in \mathbb{R}, A^i_j, B^i_j \in \mathbb{R}^6 \) and

\[
\varphi_j(\xi) = A^i_0 + \sum_{l=0}^{N_f} \left( A^i_l \cos l\xi + B^i_l \sin l\xi \right).
\]

2.7 Computation of a torus

The tori we are looking for are embedded in 2–parametric families, which can be parametrized by 2 parameters among \( h, \rho, \delta \). Therefore, in order to compute a torus, we

- eliminate one coordinate of \( A^0_0 \), in order to fixate a curve on the torus,
- set a coordinate of \( A^0_1 \) equal to zero and eliminate it, in order to get rid of the second indetermination
- eliminate two coordinates among \( h, \delta, \rho \), in order to fixate a particular torus.

When applying Newton’s method, we end up with a \((1 + 6m(1 + 2N_f)) \times (3 + 6m(1 + 2N_f) - 4)\) system of linear equations, which has more equations than unknowns. This is not a problem, as long as we use one of the general routines described in section ?? (by specifying the kernel dimension to be zero, if necessary).
2.7.1 Error estimation

In order to estimate the error of the computed torus, we can consider a refinement of the discretization of the parameter space, that is,

\[ \tilde{\xi}_j = j \frac{2\pi}{M}, \]

for \( M \gg 1 + 2N_f \), and use as error estimate

\[
\max_{j=0:M} \left\| \left( \begin{array}{c} \varphi_{l+1}(\tilde{\xi}_j) - \phi_{\delta/m}(\varphi_l(\tilde{\xi}_j)) \\ \varphi_0(\tilde{\xi}_j + \rho) - \phi_{\delta/m}(\varphi_{m-1}(\tilde{\xi}_j)) \end{array} \right)_{l=0}^{m-2} \right\|
\]

for some norm.

We can reduce this estimate by increasing \( N_f \), but this increases the size of the linear system we need to solve, and this is the bottleneck of the procedure.

2.7.2 Globalization of a torus from an invariant curve

Assume we have \( \varphi \) satisfying \( \varphi(\xi + \rho) = \phi_{\delta}(\varphi(\xi)) \). Then a calculation shows that

\[
\psi(\theta_1, \theta_2) = \phi_{\frac{2\pi}{\rho}} \left( \varphi(\theta_1 + \frac{2\pi}{\rho}) \right)
\]
describes an invariant torus with frequency vector \((\rho/\delta, 2\pi/\delta)\), that is,

\[
\phi_t(\psi(\theta_1, \theta_2)) = \psi \left( \theta_1 + t \frac{\rho}{\delta}, \theta_2 + \frac{2\pi}{\delta} \right).
\]

If we need to integrate a trajectory on the computed torus for a long time, we just have to numerically integrate it from invariant curve to invariant curve.

For instance, assume we want to generate 400 points along 4\( \delta \) time units, starting from \( x_0 = \varphi(\xi_0) \), and assume that there is no multiple shooting. Then, we can do:

\[
\forall i = 1 \div 100
\]

\[
x_i+1 := \text{RKFlow}(\delta/100, x_i, \text{tol}, f)
\]

at this point \( x_{100} \) should be \( \varphi(\xi_0 + \delta) \)

print \( \| \varphi(\xi_0 + \rho) - x_{100} \| \) as a measure of precision

\[
x_{100} := \varphi(\xi_0 + \rho)
\]

\[
\forall i = 101 \div 200
\]

\[
x_i+1 := \text{RKFlow}(\delta/100, x_i, \text{tol}, f)
\]

at this point \( x_{200} \) should be \( \varphi(\xi_0 + 2\rho) \)

etc.
2.7.3 Starting from a periodic orbit

Let \( x_0 \) be an initial condition of a \( T \)-periodic orbit, with stability parameters \( s_1, s_2 \). Assume that it has central part,

\[
|s_1| < 1, \quad s_1 =: \lambda_1 + \lambda_1^{-1}, \quad \lambda_1 = \cos \nu + i \sin \nu.
\]

Assume that \( v_1 + iv_2 \) is an eigenvector of the monodromy matrix \( M := D\phi_T(x_0) \), corresponding to the \( \lambda_1 \) eigenvalue. Then \( x_0 \) is a fixed point of \( \phi_T := F \) with multiplier \( \lambda \).

Recall that we have shown that, for

\[
\varphi(\xi) := x_0 + [v_1, v_2]R_{\xi-\xi_0} \begin{pmatrix} \gamma \\ 0 \end{pmatrix}
\]

\[
= x_0 + \gamma \left( (v_1 \cos(\xi - \xi_0) - v_2 \sin(\xi - \xi_0))v_1 \right)
\]

we have

\[
L_{\phi_T}(\varphi(\xi)) = \varphi(\xi + \nu).
\]

Therefore, as initial seed to get a torus around the o.p., we can take

\[
h = H(x_0), \quad A_0 = x_0,
\]

\[
\delta = T, \quad A_1 = (v_1 \cos \xi_0 - v_2 \sin \xi_0),
\]

\[
\rho = \nu, \quad B_1 = (v_1 \sin \xi_0 + v_2 \cos \xi_0),
\]

\[
A_j = B_j = 0, \quad j \geq 2.
\]

Notes:

- We can use \( \xi_0 \) to get one coordinate of \( A_1 \) equal to zero and, in this way, avoid the second indetermination.

- For the nonlinear system \( \rho \neq \nu \), but we don’t know if either \( \rho > \nu \) or \( \rho < \nu \), but we don’t know which. The same happens with \( T \).

- The o.p. itself satisfies the equations of an invariant torus, and has a large basin of attraction as a zero of these equations.

We can avoid the problems of the last two points above at once by keeping constant one coordinate of \( A_1 \) or \( B_1 \) which is different from zero in the initial seed. For instance, the ”opposite” coordinate to the one used to get rid of the second indetermination.

2.7.4 Starting from a periodic orbit, second method

We have seen that

\[
L_\varphi(\xi) = x_0 + \gamma[v_1, v_2]R_{\xi-\xi_0} \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]
parametrizes a closed curve invariant by the linearized time–T flow. We can globalize this invariant curve to a 2D torus invariant by the linearized flow as

\[ L_{\psi}(\theta_1, \theta_2) = \phi_{\theta_1/(2\pi)}(x_0) + D\phi_{\theta_2/2\pi}(L_{\psi}(\xi) - x_0) \]

Let us denote, for an arbitrary function \( G \) and an arbitrary point \( y_0 \) the linearization of \( F \) around \( y_0 \) as

\[ L_{y_0}^G(y) = y_0 + DG(y_0)(y - y_0). \]

Then, a calculation shows that

\[ L_{\phi_{\theta_2/(2\pi)}(x_0)}^\psi(\theta_1, \theta_2)) = L_{\psi}(\theta_1 + t\nu T, \theta_2 + t\frac{2\pi}{T}). \]

The previous way to obtain an initial seed to compute a torus corresponds to take \( \delta \) close to the second frequency of the linear torus (which corresponds to the period of the backbone periodic orbit). In some situations, we will want to get an initial seed for a torus with \( \delta \) close to the first frequency of the linear torus (a normal frequency to the backbone p.o.). For that, we can take as initial seed

\[ h = H(x_0), \quad \delta = \frac{2\pi}{\nu} T, \quad \rho = \frac{2\pi}{T}. \]

and \( A_i, B_j \) coming from a Discrete Fourier transform of \( \{L_{\psi}(0, j\frac{2\pi}{N})\}_{j=0}^{N-1} \).

\section{3 Continuation of objects}

Up to now, we have seen how to compute individual o.p. and tori. But in autonomous Hamiltonian systems, neither o.p. nor tori are isolated but embedded in families, so we will be interested in following a family. Another common situation is to continue an object (which could be isolated or not) along a family of models. We will learn in this section how we can do that.

\subsection{3.1 The continuation method}

A classical way to introduce the continuation method is the following. Assume we want to find a zero of a function \( G(x) \), which is unknown, but by some reason we know that it should be close to a known zero \( x_0 \) of another function \( F(x) \). An example of this situation is to find the dynamical substitutes of the libration points of the RTBP in the BCP.

To look for a zero of \( G \), we can consider a one–parametric family of intermediate problems \( H(\lambda, x) \), such that \( H(0, x) = F(x) \) and \( H(1, x) = G(x) \), for instance,

\[ H(\lambda, x) = (1 - \lambda)F(x) + \lambda G(x), \]

(it is called a \textit{convex homothopy} between \( F \) and \( G \)). Then we can try to continue the solution \( x_0 \) of \( H(0, x) = 0 \) up to a solution of \( H(1, x) = 0 \) with respect to the parameter \( \lambda \). A “natural” way to do this is given by the following algorithm:
input: \( x_0 \in \mathbb{R}^n \) such that \( H(0, x_0) = 0 \)
do: \[ \Delta \lambda := 1 / m \]
\[ \forall i = 1 \div m \]
\[ \lambda := i \Delta \lambda \]
solve \( H(\lambda, y) = 0 \) iteratively for \( y \) taking \( x \) as starting value
output: \( x \)

This procedure breaks down if there is a turning point along the continuation curve.

### 3.2 The predictor–corrector or pseudo–arclength method

In order to be able to continue a curve which has turning points, we can use the **predictor–corrector** or **pseudo–arclength** continuation method.

It can be described as follows. Define \( y = (\lambda, x)^\top \in \mathbb{R}^{n+1} \). Then \( H(y) := H(\lambda, x) \) defines implicitly a curve in \( \mathbb{R}^{n+1} \) (as long as rank \( DH(y) = n \), which we will assume). Then we can continue it as follows:

input: \( y \in \mathbb{R}^n \) such that \( H(y) = 0 \)
do: \[ \text{while } (\lambda = \Pi_0 y < 1) \]
\[ \text{let } v \in \ker DH(y), \|v\|_2 = 1, \text{ pointing in the right direction} \]
\[ \text{take } z := y + \gamma v, \text{ for suitable } \gamma \]
\[ \text{if } (\Pi_0 y < 1) \]
\[ \text{solve } H(z) = 0 \text{ iteratively for } z \text{ by Newton’s method taking minimum–norm corrections} \]
else
\[ \gamma := (1 - \Pi_0 y) / \Pi_0 v \]
\[ z := y + \gamma v \]
\[ \text{solve } H(z) = 0 \text{ by Newton keeping } \Pi_0 z \text{ constant} \]
output: \( y \)

In this algorithm, \( \gamma \) should be chosen in order to keep (more or less) constant the number of Newton iterates in the refinement phase. A simple rule to do that is to assume that the number of Newton iterates is a linear function of the steplength chosen. In this way, if in some iteration of the loop \( n_{old} \) is the number of Newton iterates performed in the last Newton refinement, \( \gamma_{old} \) is the last steplength used and \( n_{des} \) is the desired number of Newton iterates, we take

\[ \gamma = \frac{n_{des}}{n_{old}} \gamma_{old}. \]

Note that in the pseudo–arclength method there is no distinguished coordinate to be thought as a parameter. We can therefore apply it to any system of non–linear equations \( H(y) = 0 \), as long as its solution is a curve (and, in particular, \( \ker DH(y) \) has dimension one at every solution point \( y \)).
3.3 Examples

3.3.1 Continuation of a family of p.o. of the RTBP

We will consider two cases:

- Continuation with respect to the energy with multiple shooting. The equations to continue are

\[
\begin{align*}
H(x_0) - h &= 0, \\
\phi_{T/m}(x_i) - x_{i+1} &= 0, \quad i = 0 \div m - 2, \\
\phi_{\tau(x_{m-1})}(x_{m-1}) - x_0 &= 0,
\end{align*}
\]

with unknowns \(h, x_0, \ldots, x_{m-1}\). Note this in this case \(T\) is a parameter, but should be close to the period of the p.o. It is convenient to set \(T := \tau(x_m)\) at every continuation step, and recompute \(x_1, \ldots, x_{n-1}\), in order to have them equally spaced in time along the p.o.

- Continuation with respect to the period with multiple shooting. The equations to consider are

\[
\begin{align*}
\tau(x_{m-1}) - \frac{T}{m} &= 0 \\
\phi_{T/m}(x_i) - x_{i+1} &= 0, \quad i = 0 \div m - 2, \\
\phi_{\tau(x_{m-1})}(x_{m-1}) - x_0 &= 0,
\end{align*}
\]

with unknowns \(T, x_0, \ldots, x_{m-1}\).

Remember that the two systems of equations just considered can be evaluated by the same routine by eliminating suitable equations and unknowns.

3.3.2 Continuation of a p.o. along a one-parametric family of models

Assume we want to compute the dynamical substitute (a periodic orbit) of the libration point \(L_4\) in the BCP. Denote the differential equations of the RTBP by

\[\dot{x} = f(x),\]

and the ones of the BCP (for the same mass parameter \(\mu = m_2/m_1\)) by

\[\dot{x} = g(t, y).\]

We can define a family of intermediate models,

\[\dot{x} = h_\varepsilon(t, x)\]

given by

\[h_\varepsilon(t, x) = (1 - \varepsilon)f(x) + \varepsilon g(t, x).\]

The \(L_4\) libration point gives a p.o. in the intermediate model for \(\varepsilon\) small enough.

We can continue this solution up to the BCP by using the pseudo-arclength method with

\[H(\varepsilon, x) = \phi_T(\varepsilon) - x,\]
where $\phi^T$ is the flow from time 0 to time $T$ of
\[ \dot{x} = h_x(t,x), \]
and $T$ is the period of the BCP.

### 3.3.3 Continuation of a family of tori in the RTBP

Remember that the corresponding equations are
\[
\begin{align*}
H(\varphi_0(0)) - h &= 0 \\
\varphi_{j+1}(\xi_i) - \phi_{\delta/m}(\varphi_j(\xi_i)) &= 0, \quad j = 0 \div m - 2, \quad i = 0, \ldots, 2Nf, \\
\varphi_0(\xi_i + \rho) - \phi_{\delta/m}(\varphi_{m-1}(\xi_i)) &= 0, \quad i = 0 \div 2Nf,
\end{align*}
\]
with unknowns $h, \delta, \rho, A_{00}, A_{10}, B_{10}, \ldots, A_{m-1, Nf}, B_{m-1, Nf}, \ldots, A_{m-1}, B_{m-1}$.

Assume $A_{01}, A_{11} = 0$ are fixed in order to eliminate indeterminations, so that each value of the remaining coordinates corresponds at most to a torus.

The tori we are looking for are embedded in two–parametric families, so we have to fix one more parameter in order to use the pseudo–arclength method. Interesting cases are

- to fix $\rho$ to a diophantine value (e.g. an integer plus the golden number), so that we can go across energy levels avoiding resonances.
- to fix $h$, in order to follow an iso–energetic family.

### 4 The center manifold of the collinear points of the RTBP

As an example, we will describe how to apply all the previous methodology to numerically extend the center manifold of the collinear libration points of the RTBP. We will focus on the $L_1$ point for the Earth–Moon mass ratio.

#### 4.1 Periodic orbits

The $L_1$ point is checked to be of the type center×center×saddle. By Liapunov’s center theorem, each center gives rise to a family of periodic orbits. The eigenplane corresponding to one of them is contained in $\{z = p_z = 0\}$, so the corresponding family p.o. is known as the planar Liapunov family. The eigenplane corresponding to the other center is contained in $\{x = p_x = y = p_y = 0\}$, so the corresponding family of p.o. is known as the vertical Liapunov family.
In figure 1 we have represented the bifurcation diagram of the vertical Liapunov family. It starts at energy $-1.59417$ (the one of $L_1$), has a bifurcation at energy $-1.49590$ (which we will comment later on) and ends at a planar orbit at energy $0.41391$.

In figure 2 we have represented the bifurcation diagram of the planar Liapunov family. It starts at energy $-1.59417$ (the one of $L_1$), has several bifurcations and ends at a collision with the Earth.

According to Hénon, the only possible kinds of bifurcation from the planar Lyapunov family to a family of three–dimensional orbits are the ones sketched in Figure 3. Types A and B correspond to a stability parameter crossing 2, whereas types C and D correspond to a stability parameter crossing $-2$ (and thus it is a period–doubling bifurcation). Although the different types are characterized in terms of the structure of the monodromy matrix, an initial condition for a bifurcated orbit can be obtained by doing a small displacement in the $z$ coordinate for types A,C,D, and in the $p_z$ coordinate for type B.

The bifurcations found for the planar Liapunov family, together with its classification according to Hénon, are given in table 1. The first bifurcation gives rise to the well–known family of halo orits. The second bifurcation originates a two–lane bridge which connects the planar Liapunov family with the vertical one (at its bifurcation already mentioned, at energy $-1.49590$). We display some orbits of this bridge in figure 4. The third one is

---

2By using regularization, we could continue the planar Liapunov family past the collision, but we are not interested in that. We want to perform a full continuation of the whole center manifold of $L_1$, and by continuing the families of invariant tori we will not get that far.
outside the range of energies reached by the continuation of invariant tori, so we will not
pursue it further.

<table>
<thead>
<tr>
<th>#bif.</th>
<th>Energy</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1.58718</td>
<td>A</td>
</tr>
<tr>
<td>2</td>
<td>-1.51070</td>
<td>B</td>
</tr>
<tr>
<td>3</td>
<td>-1.47464</td>
<td>C</td>
</tr>
</tbody>
</table>

Table 1: Bifurcations of the planar Liapunov family

4.2 Invariant tori

We first start from the vertical family at constant rotation number. We observe that each
family ends at a vertical Lyapunov orbit of higher energy level. We can plot each tori
computed in the \((h, \rho)\) plane. In this same diagram, we can plot the vertical Liapunov
family of p.o. as a curve with coordinates \((h, \nu)\), where \(h\) is its energy level and \(\nu \in [0, \pi]\)
such that \(2 \cos \nu\) is the stability parameter that gives central part. This gives the curve
of figure 5 from point 2 to point 3, and all the region within the curve with \(\rho > \rho(B)\).

If we continue a isoenergetic family of tori starting from a vertical orbit, at an energy
before the bifurcation of the halo family of p.o., we get the tori in figure 6, which ends at a
planar Lyapunov orbit. The limiting value of \(\rho\) of this orbit is positive, and is numerically
checked to correspond to a normal frequency of the planar orbit, concretely

\[
\rho \rightarrow \frac{(2\pi)^2}{2\pi - \nu} - 2\pi,
\]

where \(2 \cos \nu\) is the stability parameter of the ending planar Liapunov orbit which gives
central part. This value of \(\rho\) corresponds to starting from the planar p.o. with \(\delta\) corre-
Figure 4: Some orbits in the two–lane bridge joining the planar Liapunov family and the vertical one.

Figure 5: Tori around the Liapunov families of p.o.
sponding to a normal frequency (equation (3)), but taking into account that \( \nu \) is determined up to its sign and an integer multiple of \( 2\pi \), and that \( \rho \) is also determined up to an integer multiple of \( 2\pi \). These values of \( \rho \) give the curve from point 1 to 2 in figure 5.

Therefore, in order to compute the remaining families of tori, and represent them in the same figure, we can continue the families of invariant tori starting from the planar Liapunov families via the second method and with constant rotation number. This gives the curve in figure 5 from point 2 to point 1. Point 1 corresponds to the bifurcation of the Halo family from the planar Liapunov one. Point 2 corresponds to the equilibrium point \( L_1 \). Point 3 corresponds to the end of the vertical Liapunov family at a planar Liapunov orbit.

![Figure 6: Sample tori of the iso–energetic family starting from the vertical Lyapunov p.o. around \( L_1 \) of energy \(-1.59\)](image)

All these computations can be performed in parallel in a Beowulf cluster, by following a master-slave scheme. The slave processors perform one continuation step of any family in each iteration. The master process distributes the families around slaves in a round–robin fashion.

We have also continued the following families of tori, all of them with constant rotation number.

- Invariant tori around halo orbits, from the bifurcation of the halo family from the planar Liapunov family up to turning point in the energy of the halo family.
Figure 7: Representation of invariant tori around halo orbits, period–triplicated halo–type orbits and period–duplicated ones. The dotted curve represents reaching the maximum number of harmonics allowed (100).

- Invariant tori around the elliptic–hyperbolic period–triplicated halo–type family of p.o., from its appearance to the energy in which the stability parameter crosses $-2$.
- Invariant tori around the elliptic–hyperbolic period–duplicated halo–type family of p.o., in an energy range analogous to the previous one.
- Invariant tori around planar Liapunov orbits, in a short energy range starting at the bifurcation of the two–lane bridge joining it with the vertical one, in order to complete the last Poincaré sections of figure 8. This continuation is not represented in figure 5.

They are represented in figure 7, in a diagram analogous to the one of figure 5.

### 4.3 Isoenergetic Poincaré sections

The structure of the whole center manifold of the libration point $L_1$ can be better visualized by isoenergetic Poincaré sections through $\{z = 0, p_z > 0\}$. They are shown in figure 8.

All the plots have a similar structure. The exterior curve in each plot is a Liapunov planar orbit of the energy level corresponding to the plot. As this orbit is planar, it is completely included in the surface of section, and is the only orbit for which this happens. The motion inside the region bounded by the Liapunov planar orbit is quasi–periodic,
Figure 8: Isoenergetic Poincaré sections of the center manifold around $L_1$, formed by all the families of p.o. and invariant tori computed. The two last plots are magnifications around the period–bifurcated and triplicated halo–type families of p.o., and the corresponding invariant tori.
except at some gaps which cannot be distinguished in the pictures. In all the plots there is a fixed point on the $x$ axis associated to the vertical Liapunov orbit.

For small energy values, the whole picture is formed by invariant curves surrounding the fixed point associated to the vertical orbit. They are associated to the intersections of the Lissajous–type trajectories around the vertical periodic orbit, whose evolution from the planar Liapunov periodic orbits to the vertical one was displayed in figure 6. At the energy level associated to the first bifurcation of the Liapunov planar family (first bifurcation of table 1) there appear the halo family of p.o. This can be seen clearly in the Poincaré map representations, since there appear two additional fixed point surrounded by invariant curves. Increasing the values of the energy, the halo family has two relevant bifurcations, by period triplication and duplication. Within the bifurcated families there are some with central part, which are surrounded by invariant tori. These tori give rise to the “island chain” structure typical of two–dimensional area–preserving maps. To display more clearly this behaviour, the last two plots of figure 8 we display a magnification of the bifurcated periodic orbits and its surrounding invariant tori.

The region between the tori around the vertical Liapunov orbit and the tori around the halo orbits is not empty, as it appears in the above figures, and it should contain, at least, the traces on the surface of section of the invariant manifold of the Liapunov planar orbit. These manifolds act as separatrices between both kinds of motion. The same thing happens between the islands of the bifurcated halo–type orbits and the tori around the halo orbits. In this case, the region between both kinds of tori is filled with the traces of the invariant manifolds of the bifurcated hyperbolic halo–type orbits. In all these boundary regions, the motion should have a chaotic behaviour.

The plots corresponding to energy $-1.507$ and $-1.4991$ have more structure. For these energy levels, the two–lane bridge between the planar and vertical Liapunov families of p.o. has already bifurcated, so the planar family has gained central part, and its p.o. are surrounded by invariant tori. The $\{z = 0\}$ sections of these tori are the most outer curves that appear in Figure 9. In it we also see as two fixed points the intersections of the two orbits of the bridge with the surface of section. The invariant manifolds of these bifurcated p.o. are the ones that must act as separatrices between the different kinds of tori for this value of the energy.

## 5 The Lindstedt-Poincaré method

We will focus in computing some libration orbits of the RTBP equations of motion centered in a collinear equilibrium point,

\[
\begin{align*}
\ddot{x} - 2\dot{y} - (1 + 2c_2)x &= \frac{\partial}{\partial x} \sum_{n\geq 3} c_n(\mu) \rho^n P_n \left( \frac{x}{\rho} \right), \\
\dot{y} + 2\dot{x} + (c_2 - 1)y &= \frac{\partial}{\partial y} \sum_{n\geq 3} c_n(\mu) \rho^n P_n \left( \frac{x}{\rho} \right), \\
\ddot{z} + c_2z &= \frac{\partial}{\partial z} \sum_{n\geq 3} c_n(\mu) \rho^n P_n \left( \frac{x}{\rho} \right),
\end{align*}
\]

(4)
Figure 9: Magnification of the Poincaré section corresponding to energy $-1.4991$.

where $c_n$ are certain constants.

5.1 Lissajous Orbits

We follow the idea of the Lindstedt-Poincaré method to look for these 2-D invariant tori as a series expansion in two frequencies, which formally satisfies the equations of motion up to a selected order (to be defined later). The procedure, as we will see, is a recursion that determines the coefficients of each order from the ones corresponding to lower orders that have been computed in previous steps.

As a starting point we need the (non trivial) librating solutions of the linear part of (4),

$$
\ddot{x} - 2\dot{y} - (1 + 2c_2)x = 0, \\
\dot{y} + 2\dot{x} + (c_2 - 1)y = 0, \\
\ddot{z} + c_2z = 0,
$$

that can be written as

$$
x(t) = \alpha \cos(\omega_0 t + \phi_1), \\
y(t) = \kappa \alpha \sin(\omega_0 t + \phi_1), \\
z(t) = \beta \cos(\nu_0 t + \phi_2),
$$

where $\omega_0 = \sqrt{\frac{2-c_2+\sqrt{9c_2^2-8c_2}}{2}}$, $\nu_0 = \sqrt{c_2}$ and $\kappa = \frac{-(\omega_0^2+1+2c_2)}{2\nu_0}$. The free parameters $\alpha$ and $\beta$, usually known as in plane and out of plane amplitudes respectively, and $\phi_1, \phi_2$, from now on called the phases, give all the librating solutions of the linear part of the equations.
We note that, due to the autonomous character of the equations, only one phase is needed. Hence, a solution is determined by three parameters, $\alpha$, $\beta$ and a phase. Nevertheless we will keep two phases because, as we will see, the computations in both cases are exactly the same.

When we consider the nonlinear terms of the equations, we look for formal expansions in powers of the amplitudes $\alpha$ and $\beta$, of the type

$$
x(t) = \sum_{i,j=1}^{\infty} \left( \sum_{|k| \leq i, |m| \leq j} x_{ijkm} \cos(k\theta_1 + m\theta_2) \right) \alpha^i \beta^j,
$$

$$
y(t) = \sum_{i,j=1}^{\infty} \left( \sum_{|k| \leq i, |m| \leq j} y_{ijkm} \sin(k\theta_1 + m\theta_2) \right) \alpha^i \beta^j,
$$

$$
z(t) = \sum_{i,j=1}^{\infty} \left( \sum_{|k| \leq i, |m| \leq j} z_{ijkm} \cos(k\theta_1 + m\theta_2) \right) \alpha^i \beta^j,
$$

where $\theta_1 = \omega t + \phi_1$ and $\theta_2 = \nu t + \phi_2$. Due to the presence of nonlinear terms the frequencies $\omega$ and $\nu$ can be kept no longer constant and they must be expanded as well in powers of the amplitudes: $\omega = \sum_{i,j=0}^{\infty} \omega_{ij} \alpha^i \beta^j$ and $\nu = \sum_{i,j=0}^{\infty} \nu_{ij} \alpha^i \beta^j$.

The goal is to compute the coefficients $x_{ijkm}$, $y_{ijkm}$, $z_{ijkm}$, $\omega_{ij}$ and $\nu_{ij}$ iteratively up to a finite order $i+j = n_f$ with the following meaning. Identifying the first coefficients of the general solution with the ones obtained from the solution of the linear part, we see that the non zero values are $x_{1010} = 1$, $y_{1010} = \kappa$, $z_{0101} = 1$, $\omega_{00} = \omega_0$ and $\nu_{00} = \nu_0$. The insertion of the expression of the solution of the linear part in (4) produces a remainder which are series in $\alpha$, $\beta$, beginning with terms of order $i+j = 2$. In what follows, it is said that the solution is computed up to order $i+j = n$ when, after inserting the obtained expression for the solution into the equations, the remainder consists of three series beginning with terms of order $i+j = n+1$.

Before going into the details of the computations we note that $x(t)$ and $z(t)$ are both written as a cosinus series and $y(t)$ as a sinus one. This can be done due to the symmetries of the problem and to the selected expression for the solution of the linear part. It is not difficult to see that if $j$ is odd the coefficients $x_{ijkm}$ and $y_{ijkm}$ are zero, and if $j$ is even the coefficients $z_{ijkm}$ are zero. As it is usual in the Lindstedt-Poincaré method, we only work with coefficients such that $|k| \leq i$ and $|m| \leq j$ and, moreover, $k$ and $m$ must have the same parity as $i$ and $j$ respectively. Due to the symmetries of sinus and cosinus, we can assume that $k \geq 0$ and, when $k = 0$, we can assume $m \geq 0$. Finally, in the expansions of the frequencies $\omega$ and $\nu$ only terms with both $i$ and $j$ even appear. All these properties can be taken into account to save computing time and storage.

At the first step of the procedure we start with the series $x(t)$, $y(t)$ and $z(t)$ determined up to order one and the series $\omega$ and $\nu$ up to order zero. At a certain step we will have the solution up to order $n-1$, this is, $x(t)$, $y(t)$ and $z(t)$ determined up to order $n-1$ and $\omega$ and $\nu$ up to order $n-2$ if $n-1$ is odd, or up to order $n-3$ if $n-1$ is even. In any case, we will say that $\omega$ and $\nu$ are determined up to order $n-2$, remembering that for odd orders the corresponding coefficients are zero.
Assume that we have computed the solution up to order \( n - 1 \). When we insert this expression in the right-hand side of the equations (4), we obtain three series determined up to order \( n \). Denote by \( p, q \) and \( r \) these series, that are respectively of the same type as \( x, y \) and \( z \). We are interested in finding the terms of order \( n \) of the solution by equating the (known) order \( n \) terms of \( p, q \) and \( r \) with the corresponding (and unknown) terms coming from the left hand side of equations (4). From now on (unless otherwise stated), and to simplify the notation, \( v_{ijkm} \) will refer to the terms with \( i + j = n \) of a series \( v \).

When \( k \) and \( m \) are omitted it will refer to a frequency type series.

Hence, the order \( n \) of the equations must be equated from both sides of (4). In the left-hand side there appear some derivatives which can be computed in the following way,

\[
\dot{x} = \frac{\partial x}{\partial \theta_1} d\theta_1 + \frac{\partial x}{\partial \theta_2} d\theta_2 = \omega \frac{\partial x}{\partial \theta_1} + \nu \frac{\partial x}{\partial \theta_2},
\]

\[
\ddot{x} = \omega^2 \frac{\partial^2 x}{\partial \theta_1^2} + 2\omega \nu \frac{\partial^2 x}{\partial \theta_1 \partial \theta_2} + \nu^2 \frac{\partial^2 x}{\partial \theta_2^2}.
\]

Similar expressions can be derived for \( \dot{y}, \ddot{y} \) and \( \dddot{z} \).

Each summand of the former expressions contains products of series of different type, for instance \( \omega \frac{\partial x}{\partial \theta_1} \) in the first order derivatives and \( \omega^2 \frac{\partial^2 x}{\partial \theta_1^2} \) in the second order ones. Since the series \( x, y \) and \( z \) are known up to order \( n - 1 \) and \( \omega \) and \( \nu \) up to order \( n - 2 \), the order \( n \) of these products of series contains a known part which can be added to \( p_{ijkm}, q_{ijkm}, r_{ijkm} \) and an unknown part that consists of products of terms which have to be determined. Let us see this in more detail.

Let \( f v \) be one of the terms that appear in the computations of the first derivative (\( f \) denotes the series of a frequency and \( v \) denotes the derivative of a coordinate series). The known part of order \( n \) of \( f v \) is obtained multiplying the terms of order \( i_f \) of \( f \) with the terms of order \( j_v \) of \( v \) such that \( i_f + j_v = n \), being \( i_f = 1, \ldots, n - 2 \). The unknown part of order \( n \) appears when multiplying the respective parts of order 0 and \( n \), and order \( n - 1 \) and 1 of \( f \) and \( v \). The following table summarizes the unknown parts of order \( n \) in the computations of the first derivatives of \( x \) and \( y \). The symbol \( \delta \) stands for Kronecker’s delta.

<table>
<thead>
<tr>
<th>( f )</th>
<th>( v )</th>
<th>( \omega \frac{\partial x}{\partial \theta_1} )</th>
<th>( \omega \frac{\partial x}{\partial \theta_2} )</th>
<th>( \omega \frac{\partial y}{\partial \theta_1} )</th>
<th>( \omega \frac{\partial y}{\partial \theta_2} )</th>
<th>( \nu \frac{\partial y}{\partial \theta_1} )</th>
<th>( \nu \frac{\partial y}{\partial \theta_2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( n )</td>
<td>( -\omega_{00} k x_{ijkm} )</td>
<td>( -\nu_{00} m x_{ijkm} )</td>
<td>( \omega_{00} k y_{ijkm} )</td>
<td>( \nu_{00} m y_{ijkm} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( n - 1 )</td>
<td>1</td>
<td>( -\omega_{i-1,j} \delta_{ik} \delta_{km} )</td>
<td>0</td>
<td>( \kappa \omega_{i-1,j} \delta_{ik} \delta_{km} )</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

If \( f \) is a frequency-type series and \( v \) a coordinate series (\( x, y \) or \( z \)), the following table summarizes the unknown parts of order \( n \) in the computations of the second derivatives.

<table>
<thead>
<tr>
<th>( f )</th>
<th>( \frac{\partial^2 f}{\partial \theta_1^2} )</th>
<th>( \frac{\partial^2 f}{\partial \theta_1 \partial \theta_2} )</th>
<th>( -2\omega \nu \frac{\partial^2 f}{\partial \theta_1 \partial \theta_2} )</th>
<th>( \nu^2 \frac{\partial^2 f}{\partial \theta_1^2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( n )</td>
<td>( -\omega_{00} k^2 v_{ijkm} )</td>
<td>( -2\omega_{00} \nu_{00} k \nu v_{ijkm} )</td>
<td>( -\nu_{00} m^2 v_{ijkm} )</td>
</tr>
<tr>
<td>( n - 1 )</td>
<td>1</td>
<td>( -f_{i-1,j} v_{1010} \delta_{ik} \delta_{lm} )</td>
<td>0</td>
<td>( -f_{ij} v_{0101} \delta_{ik} \delta_{lm} )</td>
</tr>
</tbody>
</table>

The known parts of order \( n \) coming from the first and second derivatives of the left-hand side of equations (4) are added to \( p_{ijkm}, q_{ijkm} \) and \( r_{ijkm} \). Let us denote by \( \tilde{p}_{ijkm}, \tilde{q}_{ijkm} \) and \( \tilde{r}_{ijkm} \) the resulting values.
Let us give some details about the computation of the terms \( f_{i-1j} \) and \( f_{ij-1} \) of the former table. First, we note that both are terms of order \( n - 1 \). They consist of an unknown part coming from the product of order zero of one series against order \( n - 1 \) of the another, plus a known part coming from the remaining products that produce order \( n - 1 \). As in the previous table \( f_{i-1j} \) represents the terms of order \( n - 1 \) of \( \omega^2 \) and \( f_{ij-1} \) represents the terms of order \( n - 1 \) of \( \nu^2 \), we can write \( f_{i-1j} = 2\nu_0\nu_{i-1j} + \Omega_{i-1j} \) and \( f_{ij-1} = 2\nu_0\nu_{ij-1} + N_{ij-1} \), where \( \Omega_{i-1j} \) and \( N_{ij-1} \) stand for the known parts of order \( n - 1 \) we have just mentioned.

So, the linear system of equations for the unknown terms of order \( n \) can be written as,

\[
\begin{align*}
- (\bar{\omega}_{km}^2 + 1 + 2c_2) x_{ijkm} - 2\bar{\omega}_{km} y_{ijkm} - 2(\omega_0 + \kappa) \omega_{i-1j} \delta_{1k} \delta_{0m} & = \bar{p}_{ijkm}, \\
-2\bar{\omega}_{km} x_{ijkm} + (c_2 - 1 - \bar{\omega}_{km}^2) y_{ijkm} - 2(\kappa \omega_0 + 1) \omega_{i-1j} \delta_{1k} \delta_{0m} & = \bar{q}_{ijkm}, \\
(c_2 - \bar{\omega}_{km}^2) z_{ijkm} - 2\nu_0 \nu_{ij-1} \delta_{0k} \delta_{1m} & = \bar{r}_{ijkm},
\end{align*}
\]

where \( \bar{\omega}_{km} = k \omega_0 + m \nu_0 \), \( \bar{p}_{ijkm} = \bar{p}_{ijkm} + \Omega_{i-1j} \delta_{1k} \delta_{0m} \), \( \bar{q}_{ijkm} = \bar{q}_{ijkm} + \kappa \Omega_{i-1j} \delta_{1k} \delta_{0m} \) and \( \bar{r}_{ijkm} = \bar{r}_{ijkm} + N_{ij-1} \delta_{0k} \delta_{1m} \).

When \((k, m) \neq (1, 0)\) and \((k, m) \neq (0, 1)\) we can solve (6) to find \( x_{ijkm}, y_{ijkm} \) and \( z_{ijkm} \).

Note that, if the frequencies of the linearized system are non-resonant, the determinant of the matrix of this system is always different from zero, although it can be very small due to the small divisors problem. As usual, this problem becomes more relevant for high orders.

When \((k, m) = (1, 0)\) (this can only happen when \( n \) is odd) the determinant of the \( x-y \) part in the first two equations of (6) is zero. So, we normalize taking \( x_{ij10} = 0 \) and we can solve for \( y_{ij10}, \omega_{i-1j} \). Finally \( z_{ij10} \) is determined solving \((c_2 - \bar{\omega}_{00}^2)z_{ij10} = \bar{p}_{ij10}\).

When \((k, m) = (0, 1)\) we can determine \( x_{ij01} \) and \( y_{ij01} \) by solving the first two equations of (6). As in the third equation the coefficient of \( z_{ij01} \) is zero, we normalize the solution taking \( z_{ij01} = 0 \) and solving \(-2\nu_0 \nu_{ij-1} = \bar{p}_{ij01} + N_{ij-1} \) for \( \nu_{ij-1} \).

### 5.2 Halo Orbits

Halo orbits are periodic orbits which bifurcate from the planar Lyapunov periodic orbits when the in plane (or intrinsic) and out of plane (or normal) frequencies are equal. This is a 1:1 resonance that appear as a consequence of the nonlinear terms of the equations and, hence, we have to look for these 1-D invariant tori as series expansion with a single frequency.

As Halo orbits are due to the nonlinear terms of the equations, they do not appear in the linearized equations (5). In order to apply the Lindstedt-Poincaré procedure we modify the equations of motion (4) by adding the product of the factors \( \Delta \) and \( z \) to the third equation,

\[
\begin{align*}
\ddot{x} - 2\dot{y} - (1 + 2c_2)x & = \frac{\partial}{\partial x} \sum_{n \geq 3} c_n(\mu) \rho^n P_n \left( \frac{x}{\rho} \right), \\
\ddot{y} + 2\dot{x} + (c_2 - 1)y & = \frac{\partial}{\partial y} \sum_{n \geq 3} c_n(\mu) \rho^n P_n \left( \frac{x}{\rho} \right),
\end{align*}
\]

(7)
\[ \ddot{z} + c_2 z = \frac{\partial}{\partial z} \sum_{n \geq 4} c_n(\mu) \rho^n P_n \left( \frac{x}{\rho} \right) + \Delta z, \]

We will look for 1-D invariant tori of these equations, with the condition \( \Delta = 0 \). In the procedure, the factor \( \Delta \) is expanded as a frequency-type series, this is, \( \Delta = \sum_{i,j=0}^{\infty} d_{ij} \alpha^i \beta^j \).

The coefficients \( d_{ij} \) will be computed iteratively.

We start looking for the (non trivial) librating solutions with one frequency of the linear part of (7),

\[
\begin{align*}
\ddot{x} - 2 \dot{y} - (1 + 2c_2)x &= 0, \\
\ddot{y} + 2 \dot{x} + (c_2 - 1)y &= 0, \\
\ddot{z} + c_2 z &= d_{00} z.
\end{align*}
\]

It is not difficult to see that they can be written as

\[
\begin{align*}
x(t) &= \alpha \cos(\omega_0 t + \phi), \\
y(t) &= \kappa \alpha \sin(\omega_0 t + \phi), \\
z(t) &= \beta \cos(\omega_0 t + \phi),
\end{align*}
\]

with \( d_{00} = c_2 - \omega_0^2 \) and where \( \omega_0 = \sqrt{\frac{2 - c_2 + \sqrt{9c_2^2 - 8c_2}}{2}}, \quad \kappa = \frac{-\omega_0^2 + 1 + 2c_2}{2\omega_0} \) and \( \phi \) is an arbitrary phase.

As in the case of Lissajous orbits, \( \alpha \) and \( \beta \) are called the in plane and out of plane amplitudes respectively. Of course, Halo orbits depend only on one frequency or one amplitude since they are 1-D invariant tori. The relationship between \( \alpha \) and \( \beta \) is contained in the condition \( \Delta = 0 \) which defines implicitly \( \alpha = \alpha(\beta) \). We note that, at the current step, Halo orbits are determined up to order 1, and \( \Delta = 0 \) is read as \( d_{00} = 0 \), showing again that there are not Halo orbits in the linear part of the equations.

When we consider the nonlinear terms of (7), we look for formal expansions in powers of the amplitudes \( \alpha \) and \( \beta \) of the type

\[
\begin{align*}
x(t) &= \sum_{i,j=1}^{\infty} \left( \sum_{|k| \leq i+j} x_{ijk} \cos(k\theta) \right) \alpha^i \beta^j, \\
y(t) &= \sum_{i,j=1}^{\infty} \left( \sum_{|k| \leq i+j} y_{ijk} \sin(k\theta) \right) \alpha^i \beta^j, \\
z(t) &= \sum_{i,j=1}^{\infty} \left( \sum_{|k| \leq i+j} z_{ijk} \cos(k\theta) \right) \alpha^i \beta^j,
\end{align*}
\]

where \( \theta = \omega t + \phi \) and, as in the case of 2-D invariant tori, the frequency \( \omega \) must be expanded as \( \omega = \sum_{i,j=0}^{\infty} \omega_{ij} \alpha^i \beta^j \). Moreover, now we have the expansion of the constraint \( \Delta = \sum_{i,j=0}^{\infty} d_{ij} \alpha^i \beta^j = 0. \)
As in the computation of Lissajous orbits, the symmetries of the problem allow to only consider terms with \( j \) even for the series \( x(t) \), \( y(t) \) and with \( j \) odd for the series \( z(t) \). Moreover, we can restrict ourselves to the case in which \( 0 \leq k \leq i + j \), having \( k \) and \( i + j \) the same parity. Finally, the frequency series \( \omega \) and \( \Delta \) only contain terms with both \( i \) and \( j \) even.

The final goal is to compute the coefficients \( x_{ijk} \), \( y_{ijk} \), \( z_{ijk} \), \( \omega_{ij} \) and \( d_{ij} \), iteratively up to a finite order \( i + j = n_f \), starting with the (pseudo) solution of the linear part whose non-zero values are \( x_{101} = 1 \), \( y_{101} = \kappa \), \( z_{011} = 1 \), \( \omega_{00} = \omega_0 \) and \( d_{00} = c_2 - \omega_0^2 \).

As it is usual in this kind of Lindstedt-Poincaré procedure, at some step we will have the series \( x(t) \), \( y(t) \) and \( z(t) \) determined up to order \( n - 1 \) and the series \( \omega \) and \( \Delta \) up to order \( n - 2 \) (we recall that when \( n - 1 \) is even, the terms of order \( n - 2 \) of these series are zero). When we insert these expansions in the right hand side of the equations (7) we obtain the series \( p_{ijk} \), \( q_{ijk} \) and \( r_{ijk} \) and \( \Delta z \) (note that the term \( \Delta z \) is not included in \( r_{ijk} \)). Of course, \( p_{ijk} \), \( q_{ijk} \) and \( r_{ijk} \) are of the same type as \( x \), \( y \) and \( z \) respectively. Note that they are determined up to order \( i + j = n \), and that the product \( \Delta z \) is determined up to order \( n - 2 \) if \( n - 1 \) is odd, or up to order \( n - 3 \) if \( n - 1 \) is even.

Next, we have to compute the part of order \( n \) of the derivatives with respect to time, \( \dot{x} \), \( \dot{y} \), \( \ddot{x} \), \( \ddot{y} \) and \( \dddot{z} \). As in the previous section, these computations involve products of series of different type such as \( \dot{x} = \omega \frac{\partial x}{\partial \theta} \) or \( \ddot{x} = \omega^2 \frac{\partial^2 x}{\partial \theta^2} \). The same statement holds for the computation of the part of order \( n \) of \( \Delta z \). As in the previous section, the part of order \( n \) of these products consists of an already known part and of an unknown one. The known part is added to the corresponding series \( p_{ijk} \), \( q_{ijk} \) or \( r_{ijk} \) to obtain the new series \( \bar{p}_{ijk} \), \( \bar{q}_{ijk} \) and \( \bar{r}_{ijk} \). The unknown terms are summarized in the following tables.

<table>
<thead>
<tr>
<th>( f )</th>
<th>( v )</th>
<th>( \omega \frac{\partial x}{\partial \theta} )</th>
<th>( \omega \frac{\partial y}{\partial \theta} )</th>
<th>( \Delta z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 )</td>
<td>( n )</td>
<td>( -\omega_{00} k x_{ijk} )</td>
<td>( \omega_{00} k y_{ijk} )</td>
<td>( d_{00} \bar{z}_{ijk} )</td>
</tr>
<tr>
<td>( n - 1 )</td>
<td>( 1 )</td>
<td>( -\omega_{1-1j} \delta_{1k} )</td>
<td>( \omega_{i1j} \kappa \delta_{1k} )</td>
<td>( d_{ij-1} \delta_{1k} )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( f )</th>
<th>( \frac{\partial^2 x}{\partial \theta^2} )</th>
<th>( \frac{\partial^2 y}{\partial \theta^2} )</th>
<th>( \frac{\partial^2 \Delta z}{\partial \theta^2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 )</td>
<td>( n )</td>
<td>( -\omega_{00}^2 k^2 x_{ijk} )</td>
<td>( -\omega_{00}^2 k^2 y_{ijk} )</td>
</tr>
<tr>
<td>( n - 1 )</td>
<td>( 1 )</td>
<td>( -f_{i-1j} \delta_{1k} )</td>
<td>( -f_{i1j} \kappa \delta_{1k} )</td>
</tr>
</tbody>
</table>

In the second table, \( f \) denotes the series corresponding to \( \omega^2 \). Note that its \( n - 1 \) order terms, \( f_{i-1j} \) and \( f_{ij-1} \), are made of known and unknown parts, \( f_{i-1j} = 2\omega_{00} \omega_{i-1j} + \Omega_{i-1j} \) and \( f_{ij-1} = 2\omega_{00} \omega_{ij-1} + \Omega_{ij-1} \), where \( \Omega_{i-1j} \) and \( \Omega_{ij-1} \) denote the known ones.

Finally, we can write the linear system of equations for the unknown terms of order \( n \),

\[
\begin{align*}
-(k^2 \omega_{00}^2 + 1 + 2c_2) x_{ijk} - 2k \omega_{00} y_{ijk} - 2(\omega_{00} + \kappa) \omega_{i-1j} \delta_{1k} & = \bar{p}_{ijk} + \Omega_{i-1j} \delta_{1k}, \\
-2k \omega_{00} x_{ijk} + (c_2 - k^2 \omega_{00}^2) y_{ijk} - 2(\kappa \omega_{00} + 1) \omega_{i-1j} \delta_{1k} & = \bar{q}_{ijk} + \kappa \Omega_{i-1j} \delta_{1k}, \\
(c_2 - k^2 \omega_{00}^2 - d_{00}) z_{ijk} - d_{ij-1} \delta_{1k} & = \bar{r}_{ijk} + \Omega_{ij-1} \delta_{1k} + 2\omega_{00} \omega_{ij-1} \delta_{1k}.
\end{align*}
\]

When \( k \neq 1 \) we can solve (8) for \( x_{ijk} \), \( y_{ijk} \) and \( z_{ijk} \). When \( k = 1 \) the determinant of the \( xy \) part and the coefficient of \( z_{ijk} \) are zero. Then we can normalize by taking \( x_{ij} = 0 \), \( z_{ij1} = 0 \), and solving the first two equations of (8) to find \( y_{ij1} \) and \( \omega_{i-1j} \). Finally, the third equation is solved to obtain \( d_{ij-1} \).
Table 2: Coefficients, up to order 3, of the Lindstedt-Poincaré expansion of the Halo orbits about $L_1$ in the Earth-Sun system (in this case, $\mu=3.04042398444176E-06$ and $\gamma=0.100109772778141E-01$).

5.3 Some results and tests on the expansions

The algorithms presented for the semi-analytical computation of the Lissajous and Halo orbits have been implemented using Fortran 77 language. As in the computation of the center manifold, commercial algebraic manipulators are much less efficient dealing with expansions, specially in the case where symmetries can be efficiently implemented in the computations. The use of an ad hoc code allows us to reach very high orders (obtaining then very accurate solutions) in a short time. In Tables 2 and 3 show the coefficients of the Halo and Lissajous expansions around $L_1$ in the Earth-Sun system, up to order 3. Typical plots of Halo orbits, vertical periodic (Lyapunov) orbits and Lissajous trajectories are presented in Figures 10, 11 and 12, respectively.

The accuracy of the expansions has been tested against numerical integration. For this purpose, initial conditions have been computed by tabulating the series expansions up to different orders. Then, these initial conditions have been integrated (numerically) during $\pi$ units of adimensional time. The final coordinates obtained from this numerical integration have been compared with the coordinates given by the series expansion at time $\pi$. If the difference in the euclidean norm for the positions is lower than $10^{-6}$ adimensional RTBP units we say that the initial condition is accurate (in the sense that it is very close to one of the periodic or quasi-periodic trajectories computed by the Lindstedt-Poincaré method). Similar results are obtained using euclidean norm with positions and velocities. The threshold has been selected to be $10^{-6}$ because, due to the instability of the orbits around the collinear points, an error of this amount in position coordinates after $\pi$ units of time, implies an error of about 100 meters in the initial conditions for the Earth-Sun...
<table>
<thead>
<tr>
<th>$i$</th>
<th>$j$</th>
<th>$\pm_{ij}$</th>
<th>$\nu_{ij}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0.2086453564223108E+01</td>
<td>0.2015210662996640E+01</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
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<td>0.2227430750989766E+00</td>
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<tr>
<td>0</td>
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<td>0.2581841437578153E-01</td>
<td>-0.1631915758176957E+00</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>$i$</th>
<th>$j$</th>
<th>$k$</th>
<th>$m$</th>
<th>$x_{ijk}$ or $z_{ijk}$</th>
<th>$y_{ijk}$</th>
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<tr>
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<td>1</td>
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<td>0.3549452858304732E+00</td>
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</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0</td>
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<tr>
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<tr>
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<td>2</td>
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<td>-2</td>
<td>-0.149994891576764E+01</td>
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<td>1</td>
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<td>2</td>
<td>1</td>
<td>0.4060793036977860E+00</td>
<td>0.1952722175104363E-01</td>
</tr>
</tbody>
</table>

Table 3: Coefficients, up to order 3, of the Lindstedt-Poincaré expansion of the Lissajous orbits about $L_1$ in the Earth-Sun system.

Figure 10: Projections on the coordinate planes and a 3D representation of a Halo orbit about $L_1$ in the Earth-Sun System with $\beta = 0.1$.
Figure 11: Projections on the coordinate planes and a 3D representation of a vertical periodic orbit about $L_1$ in the Earth-Sun System and with $\alpha = 0.0$ and $\beta = 0.1$.

Figure 12: Projections on the coordinate planes and a 3D representation of a Lissajous orbit about $L_1$ in the Earth-Sun System and with $\alpha = 0.05$ and $\beta = 0.15$. 
system. This is because errors increase by a factor close to 1500 after π units of time due to the hyperbolic character of the orbits. Moreover, the use of different thresholds produces qualitatively similar results.

In this way, given a couple of amplitudes \((\alpha, \beta)\), we can check if a mesh of points (initial conditions) on the corresponding Lindstedt-Poincaré series satisfy the accuracy criterion given above. In that case, we say that \((\alpha, \beta)\) belongs to the domain of practical convergence. Here, “practical” means that the series looks convergent for numerical purposes (in fact, it is well know that those series are divergent due to the effect of the small divisors).

The regions of practical convergence for the case of Lissajous orbits are represented in Figure 13, using as example the expansion around \(L_1\) for the Earth-Sun system. Each continuous line represents the boundary of the domain of practical convergence for a selected order of truncation of the expansion. Given any couple \((\alpha, \beta)\) inside the region of practical convergence, the error after the numerical integration is less than the selected threshold for all the phases corresponding to the different initial conditions. A similar plot is used and discussed in [5].

The same test has been done for the case of Halo orbits. In this case the expansion only depends on one amplitude. The maximum \(\beta\) amplitude reached for some orders is given in Table 4.

<table>
<thead>
<tr>
<th>order</th>
<th>(\beta) ampl.</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>0.196</td>
</tr>
<tr>
<td>15</td>
<td>0.336</td>
</tr>
<tr>
<td>21</td>
<td>0.461</td>
</tr>
<tr>
<td>25</td>
<td>0.502</td>
</tr>
<tr>
<td>31</td>
<td>0.557</td>
</tr>
<tr>
<td>35</td>
<td>0.604</td>
</tr>
</tbody>
</table>

Table 4: Maximum \(\beta\) amplitude which gives the region of convergence using the criterion explained in the text, for the case of Halo orbits around \(L_1\) in the Earth-Sun system.

6 Lindstedt Poincaré about periodic orbits

Let us consider again the RTBP and the related equations of motion (4). Denoting by \(x = (x_1, x_2, x_3, x_4, x_5, x_6) \equiv (x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3)\), the equations of motion can be written in compact form as \(\dot{x} = f(x)\). Assume now that \(\bar{x}(t) = (\bar{x}_1(t), \bar{x}_2(t), \bar{x}_3(t), \dot{\bar{x}}_1(t), \dot{\bar{x}}_2(t), \dot{\bar{x}}_3(t))\) is a halo orbit, solution of the equations (4). The time dependent change of variables \(x = z + \bar{x}\) transforms \(\dot{x} = f(x)\) into \(\dot{z} + \dot{\bar{x}} = f(z + \bar{x})\). Expanding \(f(z + \bar{x})\) in power series around the halo orbit we get

\[
\begin{align*}
    f(z + \bar{x}) &= f(\bar{x}) + A(\bar{x})z + g(z, \bar{x}),
\end{align*}
\]
where $A(\bar{x})$ is a time dependent matrix, and $g(z, \bar{x})$ is a function whose Taylor series expansion in powers of $z_i$, $i = 1 \ldots 6$, starts at order two and it has time dependent coefficients. Both $A$ and $g$ can be obtained by symbolic manipulation.

Since the halo orbit is a solution of the initial equations, we have $\dot{\bar{x}} = f(\bar{x})$, and so

$$\dot{z} = A(\bar{x}(t))z + g(z, \bar{x}(t)), \quad (9)$$

where the linear part of these equations, $\dot{z} = A(\bar{x}(t))z$, gives rise to the variational equations on the halo orbit.

The fact that the halo orbit is periodic implies that the time dependence in (9) is also periodic. In this way, the halo orbit $\bar{x}(t)$ can be seen as a function of an angle $\Phi_1$, $\bar{x}(\Phi_1)$, where $\Phi_1 = \omega t + \phi_1$, $\omega$ is the frequency of the halo orbit, and $\phi_1$ is a phase. If $T$ denotes the period of the halo orbit then $T \omega = 2\pi$.

In order to apply the LP procedure, it is advisable that the matrix $A$ of (9) be constant. As its time dependence is periodic, it can be removed via a Floquet change of coordinates. This is a linear change of variables with periodic coefficients that we will denote by $z = Py$, where $P = P(\Phi_1)$ is a periodic matrix.

Introducing $z = Py$ in (9) we get $\dot{P}y + P\dot{y} = APy + g(Py, \Phi_1)$, that is,

$$\dot{y} = (P^{-1}AP - P^{-1}\dot{P})y + P^{-1}g(Py, \Phi_1).$$

The matrix $P$ is computed solving $\dot{P} = AP - PB$, where $B$ is known as the Floquet matrix. $B$ is a constant matrix which is a logarithm of the monodromy matrix of $\dot{z} = A(\bar{x}(t))z$ divided by the period $T$ (see [4]). In this way, equations (9) become

$$\dot{y} = By + Qg(Py, \Phi_1), \quad (10)$$

where $Q = P^{-1}$. In the present problem, the matrix $B$, after a suitable change of basis, can be simplified to

$$B = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
b_{41} & 0 & b_{43} & 0 & b_{45} & 0 \\
0 & 0 & 0 & b_{54} & 0 & b_{56} \\
b_{61} & 0 & b_{63} & 0 & b_{65} & 0
\end{pmatrix}. \quad (11)$$

Note that the final equations (10) can be obtained from the original ones (4) using the linear change of variables

$$x = P(\Phi_1)y + \bar{x}(\Phi_1). \quad (12)$$

Next we are going to show the main points of the computation and implementation of this transformation.

### 6.1 Computational aspects

In order to get equation (10), we must know the matrix $P(\Phi_1)$, the halo orbit $\bar{x}(\Phi_1)$, and the constant Floquet matrix $B$.
Taking into account that the final goal of the procedure is to obtain Lissajous orbits around a halo orbit expanded as a series, and that the time dependence we are dealing with is periodic, all the periodic functions and coefficients are handled using Fourier series up to a certain order. Initially, one must compute the halo orbit. This can be done either numerically or semianalytically. Both procedures will be discussed in what follows.

One can compute a halo orbit using a numerical procedure for computing families of periodic orbits and then to perform a Fourier analysis of its coordinates. We can also use the Lindstedt–Poincaré procedure for the computation of the family of halo orbits and then determine the desired orbit in the family. Both approaches give us the halo orbit as Fourier series that can be chosen of cosinus type for the $x_1$ and $x_3$ components and of sinus type for $x_2$, due to the symmetries of the problem. This is,

\[
\bar{x}_1 = \sum_k \bar{x}_{1k} \cos k \Phi_1, \quad \bar{x}_2 = \sum_k \bar{x}_{2k} \sin k \Phi_1, \quad \bar{x}_3 = \sum_k \bar{x}_{3k} \cos k \Phi_1.
\]

The next step is to compute the matrix $B$. This is somewhat longer. As we have said, the matrix $B$ is a logarithm of the monodromy matrix around the halo orbit divided by the period $T$ of the halo orbit. The monodromy matrix can be obtained numerically integrating the variational equations on the halo orbit at the same time that we compute the halo orbit numerically. In case that the Fourier expansions for the coordinates of a halo orbit are already known, for instance if we use a LP procedure for the halo orbits, it can be obtained as well integrating only the variational equations $\dot{Z} = A(\Phi_1) Z$, where $A(\Phi)$ can be computed using these expansions. Of course $Z(0) = I$. This second approach is more cumbersome to carry out in practice since the semianalytical computation of the matrix $A(\Phi_1)$ must be done using both the change of coordinates, $x = z + \bar{x}$, and the truncated Fourier series for the halo orbit. Using symbolic manipulation, the linear part of the resulting equations, which gives us $A(\Phi_1)$, can be solved. In order to check the results and to compute them efficiently, we use the fact that, due to the symmetries of the equations, the matrix $A$ has its entries, $a_{ij}(\Phi_1)$, of cosinus (resp. sinus) type when $i + j$ is even (resp. odd).

Assume that the monodromy matrix, $M = Z(T)$, is available. To compute its logarithm, first we put it in Jordan form via a change of basis $V$. This is, $M_J = V^{-1} M V$, where for small halo orbits the structure that we have is

\[
M_J = \begin{pmatrix}
\lambda & 0 & 0 \\
\lambda^{-1} & 1 & \varepsilon \\
0 & 0 & 1 \\
0 & \cos \Gamma & -\sin \Gamma \\
0 & \sin \Gamma & \cos \Gamma
\end{pmatrix}.
\]
The logarithm of this matrix divided by the period is

\[ F_J = \frac{1}{T} \begin{pmatrix} \log \lambda & -\log \lambda & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & -\Gamma - \end{pmatrix}, \]

and the Floquet matrix, \( F \), is obtained undoing the change \( V \), so \( F = (f_{ij}) = VFJV^{-1} \).

In order to put the Floquet matrix in the simplified form, \( B \), that appears in (11), we must perform another change of variables, \( S \), so \( B = SFS^{-1} \). The explicit expression of \( S \) in terms of the components of \( F = (f_{ij}) \) is

\[ S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & f_{12} & 0 & f_{14} & 0 & f_{16} \\ f_{21} & 0 & f_{23} & 0 & f_{25} & 0 \\ 0 & f_{32} & 0 & f_{34} & 0 & f_{36} \end{pmatrix}. \]

Finally, the Floquet change of basis is obtained by integrating \( \dot{P} = AP - \bar{P}F \), from \( t = 0 \), until \( t = T \) and taking \( \bar{P}(0) = I \). The entries of \( \bar{P} \) are periodic functions of period \( T \). To get the periodic matrix \( P \) appearing in (12), which gives the final equations (10), \( P = \bar{P}S^{-1} \) and its inverse \( Q = P^{-1} = S\bar{P}^{-1} \) are stored in a mesh of points in order to perform a Fourier analysis of its components. Due to symmetry, the respective components \( p_{ij} \) and \( q_{ij} \) of \( P \) and \( Q \) are of cosinus (resp. sinus) type when \( i + j \) is even (resp. odd).

6.2 The Lindstedt Poincaré expansion

In this section we are going to explain how we compute the quasihalo orbits of the equations (10) by means of a LP method.

First of all, we note that the equations \( \dot{x} = f(x) \) of the RTBP at the equilibrium point have \( f_1 = f_2 = f_3 = 0 \) and so in (10), \( g_1 = g_2 = g_3 = 0 \). The system (10) can be written as

\[
\begin{pmatrix}
\dot{y}_1 \\
\dot{y}_2 \\
\dot{y}_3 \\
\dot{y}_4 \\
\dot{y}_5 \\
\dot{y}_6 \\
\end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\
0 & b_{41} & 0 \\
0 & b_{51} & b_{54} \\
0 & b_{61} & b_{63} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix} \begin{pmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4 \\
y_5 \\
y_6 \\
\end{pmatrix} + Q \begin{pmatrix} 0 \\
0 \\
0 \\
g_4(Py) \\
g_5(Py) \\
g_6(Py) \\
\end{pmatrix},
\]

where the functions \( g \) depend periodically on time (both directly and through \( P \)).
Using the first three equations we obtain,
\[ \dot{y}_j = y_{j+3} + \sum q_{j,i+3}g_{i+3}(Py), \quad j = 1, 2, 3, \tag{14} \]
where in all the sums, from now on and unless otherwise stated, \( i \) runs from 1 to 3. Taking derivatives with respect to time we have, for \( j = 1, 2, 3, \)
\[ \ddot{y}_{j+3} = \dot{y}_j - \sum q_{j,i+3}g_{i+3}(Py) - \sum q_{j,i+3}\dot{g}_{i+3}(Py). \]
Inserting these derivatives into the last three equations of (13) and making use of (14),

\[ \begin{align*}
\dot{y}_1 - b_{41}y_1 - b_{43}y_3 - b_{45}\dot{y}_2 &= -b_{45} \sum q_{2,i+3}g_{i+3}(Py) + \\
\dot{y}_2 - b_{54}\dot{y}_1 - b_{56}\dot{y}_3 &= - \sum (b_{54}q_{1,i+3} + b_{56}q_{3,i+3})g_{i+3}(Py) + \\
\dot{y}_3 - b_{61}y_1 - b_{63}y_3 - b_{65}\dot{y}_2 &= -b_{65} \sum q_{2,i+3}g_{i+3}(Py) + \\
& \quad \sum (q_{3,i+3}g_{i+3}(Py) + q_{5,i+3}\dot{g}_{i+3}(Py) + q_{6,i+3}\dot{g}_{i+3}(Py))
\end{align*} \tag{15} \]

where the left hand side contains the linear part and the right hand one the higher order terms. Note that in these equations \( Py \), which appears in the functions \( g_j \), still contains the variables \( y_4, y_5, y_6 \), implying that equations (15) are coupled with equations (14).

Since the goal of a LP procedure is to get a formal series solution of equations (10), we can proceed doing the following steps, that will be explained later in more detail. Here, \textit{“order”} denotes the order with respect to the parameter \( \gamma \).

- Start by computing the solution of the linear part of (15). As we will see, this gives us the order one of the series expansion of \( y_1, y_2, y_3 \).

- Use (14) to determine \( y_4, y_5, y_6 \) up to order one. Up to this order we have \( y_4 = \dot{y}_1, y_5 = \dot{y}_2, y_6 = \dot{y}_3 \).

- Assume that \( y_1, y_2, y_3, y_4, y_5, y_6 \) have been determined up to a certain order \( n - 1 \).

- Take (15) and apply the LP procedure to determine the order \( n \) of \( y_1, y_2, y_3 \). We note that since the right hand side part of (15) begins with quadratic terms in \( Py \), for this step we only need \( y_1, y_2, y_3, y_4, y_5, y_6 \) up to order \( n - 1 \), to equate the terms of order \( n \) of both sides.

- Use (14) again to determine the terms of order \( n \) of \( y_4, y_5, y_6 \). For this purpose we need \( y_1, y_2, y_3 \) up to order \( n \), computed previously, and \( y_4, y_5, y_6 \) up to order \( n - 1 \), computed in the preceding iteration.

- Repeat the last three steps up to a desired order \( N \) (to be defined later).
Taking into account the type of series (sinus or cosinus) of the halo orbit $\bar{x}$ that determine the type of the entries of $P$ and relation (12), we get that $y_1$, $y_3$ and $y_5$ will be cosinus type series whilst $y_2$, $y_4$ and $y_6$ will be of sinus type. Then series expansions for $y_j$, $j = 1, \ldots, 6$, are searched in the following way,

$$y_j(t) = \sum_{i=1}^{\infty} \left( \sum_{k,m} y_{j,k}^{m} \cos \left( k\Phi_1 + m\Phi_2 \right) \right) \gamma^i. \quad (16)$$

For the angles $\Phi_1$ and $\Phi_2$ we use the expressions,

$$\Phi_1 = \omega t + \phi_1, \quad \Phi_2 = \nu t + \phi_2,$$

where $\Phi_1$ is the same as in the halo orbit, this is, $\omega$ is the frequency of the backbone halo orbit. The frequency $\nu$ is related to the motion of the Lissajous orbit around the halo one. Due to the nonlinear character of equations (13), it cannot be kept fixed. Specifically, $\nu$ depends on the amplitude $\gamma$ which essentially measures the displacement from the quasihalo orbit to the selected base halo one. It must be expanded in the form

$$\nu = \sum_{i=0}^{\infty} \nu_i \gamma^i. \quad (17)$$

Giving arbitrary values to the phases $\phi_1$, $\phi_2$ and to the amplitude $\gamma$ (this one inside the range of “convergence” which will be discussed later), we obtain all the quasihalo orbits about a halo orbit which have one of the fundamental frequencies equal to the one of the halo selected. At this point we notice that $\gamma = 0$ implies $y = 0$ and due to (12), the solution that we get is the halo orbit. Moreover, due to the autonomous character of the RTBP, one can fix one of the phases equal to zero and, by varying $\gamma$ and the other phase, the same set of solutions is obtained.

Assuming the halo orbit expanded as an infinite Fourier series, the range for $k$ in (16) goes from 0 to $\infty$. During the procedure only $y$ terms with $|m| \leq i$ and $m \equiv i \pmod{2}$ show up. Also only terms $\nu_i$ with $i$ even appear. Since the expansions of the halo orbit are truncated to $l$ harmonics, only terms with $k \leq l$ are kept in the expansions. Moreover when $k = 0$, changing the sign of the coefficient if necessary, only terms with $m \geq 0$ are kept. All these facts have been taken into account in the semianalytical implementation in order to save computing time and storage.

Of course the sum with respect to $i$ in (16) must be truncated at some value $N$, $(N \leq l)$, which will be known as the order of the final expansion. At the $n$-th step of the procedure the terms with $i = n$ of the $y$ variables are determined. We say that at this step we have the solution up to order $n$, meaning that if we insert formally the expressions that we have for $y$ into (10), the expansions obtained for the residuals, which are of the same type as (16), are zero for all the terms with $k \leq l$ and $i \leq n$. This property is the first one to be checked once we have computed the approximate solution at order $N$.

### 6.3 Computation of the linear part

The linear part of (15) is

$$\ddot{y}_1 - b_{41} y_1 - b_{43} y_3 - b_{45} \dot{y}_2 = 0,$$
\[ y_2 - b_{54}y_1 - b_{56}y_3 = 0, \]
\[ y_3 - b_{61}y_1 - b_{63}y_3 - b_{65}y_2 = 0. \]  

(18)

A non trivial librating solution of these equations must be used to determine the coefficients of (16) with \( i = 1 \). This is, we are looking for the expansions up to order one of \( y_j, j = 1, 2, 3 \), that we denote by \( y_j^1 \). They are,

\[ y_j^1(t) = \gamma \sum_{k,m} y_{j,1}^{k,m} \cos(k\Phi_1 + m\Phi_2). \]

Since (18) does not depend on the halo frequency \( \omega \), we must take \( y_{j,1}^{k,m} = 0, j = 1, 2, 3 \), for \( k \neq 0 \). Taking into account the above mentioned restrictions on the index, only \( y_{j,1}^{0,1} \), \( j = 1, 2, 3 \), remain as free coefficients. So, for all \( \gamma \) and \( \phi_2 \), we must look for a non trivial solution of the form

\[ y_j^1(t) = \gamma y_{j,1}^{0,1} \cos(\nu t + \phi_2). \]

When we require that these expressions satisfy (18) we find

\[ \begin{pmatrix} - (b_{41} + \nu^2) & - b_{45} \nu & - b_{43} \\ b_{54} \nu & - \nu^2 & b_{56} \nu \\ - b_{61} & - b_{65} \nu & -(b_{63} + \nu^2) \end{pmatrix} \begin{pmatrix} y_{1,1}^{0,1} \\ y_{2,1}^{0,1} \\ y_{3,1}^{0,1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]

Since we need a non trivial solution, we must select the value of \( \nu \neq 0 \) making the determinant of the former linear system equal to zero. This is easily found by solving,

\[ \nu^4 + (b_{65}b_{56} + b_{54}b_{45} + b_{41} + b_{63})\nu^2 + 2b_{41}b_{63} - b_{43}b_{61} + b_{65}b_{41} + b_{54}b_{45}b_{63} - b_{54}b_{65}b_{43} - b_{61}b_{45}b_{56} = 0. \]

The root of this equation gives us the term \( \nu_0 \) in the expansion of \( \nu \) given by (17). (Remark: The value of \( \nu_0 \) is equal to \( \Gamma/T \) where \( \Gamma \) is the value that appears in the matrix \( M_J \) and \( T \) is the period of the halo orbit). As the linear system is homogeneous, we have a free parameter in the determination of the coefficients \( y_{j,1}^{0,1}, j = 1, 2, 3 \). This freedom is related to the different possible “normalizations” in the definition of the parameter \( \gamma \). We have selected \( \gamma \) to be the coefficient of \( \cos(\nu t + \phi_2) \) in the Fourier expansion of \( y_3 \). This means that \( y_{3,1}^{0,1} = 1 \), and in this way we have an unique solution for the linear part.

Concerning the higher order terms, this choice implies \( y_{3,n}^{0,1} = 0 \) for all \( n > 1 \). Any other choice can be reduced to this one by a suitable redefinition of \( \gamma \), a fact which is assured by using the implicit function theorem.

Now we have the expansions of \( y_1, y_2 \) and \( y_3 \) up to order one,

\[ y_1^1(t) = \gamma y_{1,1}^{0,1} \cos(\nu_0 t + \phi_2), \quad y_2^1(t) = \gamma y_{2,1}^{0,1} \sin(\nu_0 t + \phi_2), \quad y_3^1(t) = \gamma y_{3,1}^{0,1} \cos(\nu_0 t + \phi_2). \]

Finally, using (14) we get the terms of \( y_4, y_5 \) and \( y_6 \) of order one. In this case we have \( y_{4,1}^{k,m} = y_{5,1}^{k,m} = y_{6,1}^{k,m} = 0 \) for \( k \neq 0 \), and \( y_{4,1}^{0,1} = -\nu_0 y_{4,1}^{0,1}, y_{5,1}^{0,1} = \nu_0 y_{2,1}^{0,1} \) and \( y_{6,1}^{0,1} = -\nu_0 y_{3,1}^{0,1} \).
6.4 Computation of the higher order terms

To compute the higher order terms we proceed recursively. Assume that at a certain step we have \( y_j, j = 1, ..., 6 \), determined up to order \( n - 1 \) and \( \nu \) up to order \( n - 2 \) (resp. \( n - 3 \)) if \( n - 1 \) is odd (resp. even) (in fact we can assume \( \nu \) determined up to order \( n - 2 \) since otherwise, when it is only determined up to order \( n - 3 \), then \( n - 2 \) is odd and \( \nu_{n-2} = 0 \)). The insertion of these known terms in the right hand side of (15) produces expansions which are completely known up to order \( n \). Let us denote them by \( \check{r}_{1,n}^k, \check{r}_{2,n}^k, \check{r}_{3,n}^{k,m} \) and \( \check{r}_{3,n}^{k,3,m} \). We note that they are series as the ones in (16), of cosinus, sinus and cosinus type respectively. Given the expansions of \( x_1, x_2 \) and \( x_3 \) up to order \( n - 1 \), they can be computed via the change of variables (12), and using the recurrences for \( T_j \) and \( R_j \) which appear in the equations (4). We note that the series for \( x_1, x_2 \) and \( x_3 \) are the final goal of our computations and they are of the same type of \( y_1, y_2 \) and \( y_3 \), respectively.

The purpose of the first part of the \( n \)-th step is to equate terms of order \( n \) in both sides of (15). Up to now we have computed the order \( n \) of the right hand side; now we must compute the order \( n \) of the left hand one. This latter part is made of an unknown part of order \( n \) and a known one that must be determined and added to \( \check{r} \).

Let us denote by \( s \) any one of the \( y \) functions. The derivatives with respect to time, which appear in the left hand side of equations (15), can be computed by

\[
\dot{s} = \frac{\partial s}{\partial \Phi_1} \frac{d\Phi_1}{dt} + \frac{\partial s}{\partial \Phi_2} \frac{d\Phi_2}{dt} = \omega \frac{\partial s}{\partial \Phi_1} + \nu \frac{\partial s}{\partial \Phi_2},
\]

\[
\ddot{s} = \omega^2 \frac{\partial^2 s}{\partial \Phi_1^2} + 2\omega \nu \frac{\partial^2 s}{\partial \Phi_1 \partial \Phi_2} + \nu^2 \frac{\partial^2 s}{\partial \Phi_2^2},
\]

where we recall that \( \omega \) is constant but \( \nu \) is given by the series (17). Since any partial derivative of \( s \) is formally a series like the ones in (16), the products of \( \nu \) by partial derivatives are series products. Moreover, \( \nu^2 \) is also a series product which is of the same type as (17). The order \( n \) of a series product that appears in our computations is obtained multiplying the part of order \( j \) of a frequency type series (17) by the part of order \( n - j \) of an expansion type series like (16) and then adding the results from \( j = 0 \) to \( j = n - 1 \) (we note that the series like (16) have no zero order part).

When \( j \) ranges from \( j = 1 \) to \( n - 2 \) in the products of \( \nu \) by a partial derivative, there appear known terms. In a similar way, known terms appear in the computation of \( \nu^2 \) which is used later when computing \( \nu^2 \frac{\partial^2 s}{\partial \Phi_2^2} \) for the same values of \( j \). Let us assume that all these values are added to the corresponding \( \check{r}_{1,n}^k, \check{r}_{2,n}^k \) and \( \check{r}_{3,n}^{k,3,m} \). When \( j = 0 \) or \( j = n - 1 \), the following table summarizes the terms that we have for \( \dot{s} \) and \( \ddot{s} \). The symbol \( \delta \) stands for Kronecker’s delta.

<table>
<thead>
<tr>
<th>( j )</th>
<th>( n - j )</th>
<th>( \dot{s} )</th>
<th>( \ddot{s} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( n )</td>
<td>( \pm (k\omega + m\nu_0) s_{n}^{k,m} )</td>
<td>( -(k\omega + m\nu_0)^2 s_{n}^{k,m} )</td>
</tr>
<tr>
<td>( n - 1 )</td>
<td>1</td>
<td>( \pm \nu_{n-1} s_{1}^{0,1} \delta_{k,0} \delta_{m,1} )</td>
<td>( -2\nu_0 \nu_{n-1} s_{1}^{0,1} \delta_{k,0} \delta_{m,1} - N_{n-1}^{2} s_{1}^{0,1} \delta_{k,0} \delta_{m,1} )</td>
</tr>
</tbody>
</table>

In the table, the \( \pm \) sign has to be taken + (resp. -) if \( s \) is a sinus (resp. cosinus) series. \( N_{n-1}^{2} \) represents the known part of \( \nu^2 \) of order \( n - 1 \), obtained by multiplying the parts
of order \( j \) of \( \nu \) with the ones of order \( n - j - 1 \) from \( j = 1 \) to \( n - 2 \). \( s_j^{k,m} \) represents the term \( (k, m) \) of the part of order \( j \) of the series \( s \).

We note that the term \( N_{n-1}^{0,1} s_1^{0,1} \delta_{k,0} \delta_{m,1} \) in the previous table is known. So, all the contributions coming from these terms in the second derivatives of the \( y \) series in the left hand side of (15) can be included in the series \( \tilde{r} \). From now on we denote by \( r_1, r_2 \) and \( r_3 \) the part of order \( n \) of the respective initial series, \( \tilde{r} \), updated with the known part of order \( n \) coming from the left hand side of equations (15). Following the usual notation, we refer to the individual terms by \( r_1^{k,m}, r_2^{k,m} \) and \( r_3^{k,m} \).

By equating the terms of order \( n \) in both sides of (15) we obtain the linear system

\[
- (\Psi^2 + b_{41}) y_{1,n}^{k,m} - b_{45} y_{2,n}^{k,m} - b_{43} y_{3,n}^{k,m} - (2\nu_0 y_{1,1}^{0,1} + b_{45} y_{2,1}^{0,1}) \nu_{n-1} \delta_{k,0} \delta_{m,1} = r_{1,n}^{k,m},
\]

\[
b_{54} y_{1,n}^{k,m} - \Psi^2 y_{2,n}^{k,m} + b_{56} y_{3,n}^{k,m} + (b_{54} y_{1,1}^{0,1} + b_{56} y_{3,1}^{0,1} - 2\nu_0 y_{2,1}^{0,1}) \nu_{n-1} \delta_{k,0} \delta_{m,1} = r_{2,n}^{k,m},
\]

\[
-b_{61} y_{1,n}^{k,m} - b_{65} \Psi y_{2,n}^{k,m} - (\Psi^2 + b_{63}) y_{3,n}^{k,m} - (2\nu_0 y_{3,1}^{0,1} + b_{65} y_{2,1}^{0,1}) \nu_{n-1} \delta_{k,0} \delta_{m,1} = r_{3,n}^{k,m},
\]

where \( \Psi = k \omega + \mu_0 \). It must be solved for the unknowns \( y_{1,n}^{k,m}, y_{2,n}^{k,m}, y_{3,n}^{k,m} \) and \( \nu_{n-1} \).

When \( (k, m) \neq (0, 1) \) it becomes

\[
\begin{pmatrix}
-b_{41} - \Psi^2 & -b_{45} & -b_{43} \\
b_{54} \Psi & -\Psi^2 & b_{56} \Psi \\
-b_{61} & -b_{65} \Psi & -b_{63} - \Psi^2
\end{pmatrix}
\begin{pmatrix}
y_{1,n}^{k,m} \\
y_{2,n}^{k,m} \\
y_{3,n}^{k,m}
\end{pmatrix}
= \begin{pmatrix}
r_{1,n}^{k,m} \\
r_{2,n}^{k,m} \\
r_{3,n}^{k,m}
\end{pmatrix},
\]

from where we get \( y_{1,n}^{k,m}, y_{2,n}^{k,m} \) and \( y_{3,n}^{k,m} \). We note that the case \( (k, m) = (0, 0) \) gives us a singular matrix, but it is easily solved taking \( y_{2,0}^{0,0} = 0 \), since \( r_2 \) is a sinus type series and so \( r_{2,0}^{0,0} = 0 \).

When \( (k, m) = (0, 1) \) (this only happens when \( n \) is odd) the matrix of (19) is singular since in the computations of the first order part we selected \( \nu_0 \) in that way. But in this case we have the additional contribution of \( \nu_{n-1} \) giving

\[
\begin{pmatrix}
-b_{41} - \nu_0^2 & -b_{45} \nu_0 & -b_{43} \\
-b_{54} \nu_0 & -\nu_0^2 & b_{56} \nu_0 \\
-b_{61} & -b_{65} \nu_0 & -b_{63} - \nu_0^2
\end{pmatrix}
\begin{pmatrix}
y_{1,n}^{0,1} \\
y_{2,n}^{0,1} \\
y_{3,n}^{0,1}
\end{pmatrix}
= \begin{pmatrix}
r_{1,n}^{0,1} \\
r_{2,n}^{0,1} \\
r_{3,n}^{0,1}
\end{pmatrix}.
\]

There is an unique value of \( \nu_{n-1} \) such that this system is solvable. Then it remains a free parameter to determine \( y_{1,n}^{0,1}, y_{2,n}^{0,1} \) and \( y_{3,n}^{0,1} \). We set \( y_{3,n}^{0,1} = 0 \), as mentioned in the discussion of the solution of the linear part, avoiding all ambiguities.

Finally we must compute the series of order \( n \) of \( y_1, y_5 \) and \( y_6 \). For this purpose we use (14). Note that for the computations we need \( y_1, y_2 \) and \( y_3 \) up to order \( n \) and \( y_4, y_5 \) and \( y_6 \) up to order \( n - 1 \).

### 6.5 Alternative procedure

We can proceed with the Lindstedt–Poincaré method working with the first order differential equations (10) instead of the second order ones. If we write down (14) explicitly,
putting the linear part in the left hand side and the nonlinear one in the right hand one, we have

\[
\begin{align*}
\dot{y}_1 - y_4 &= \sum q_{1i+3}g_{i+3}(Py), \\
\dot{y}_2 - y_5 &= \sum q_{2i+3}g_{i+3}(Py), \\
\dot{y}_3 - y_6 &= \sum q_{3i+3}g_{i+3}(Py), \\
\dot{y}_4 - b_{41}y_1 - b_{43}y_3 - b_{45}y_5 &= \sum q_{4i+3}g_{i+3}(Py), \\
\dot{y}_5 - b_{54}y_4 - b_{56}y_6 &= \sum q_{5i+3}g_{i+3}(Py), \\
\dot{y}_6 - b_{61}y_1 - b_{63}y_3 - b_{65}y_5 &= \sum q_{6i+3}g_{i+3}(Py),
\end{align*}
\]

where \(i = 1, 2, 3\) in the summations. The series expansions for the functions \(y\) are the same ones as in the previous section. The derivatives which appear in the Lindstedt–Poincaré procedure are computed in the same way as before, but now only the first order derivatives appear.

Once the linear part is solved, at each step we must solve for the unknown part of order \(n\). Proceeding in an analogous way as in the previous section we get the linear system of equations,

\[
\begin{align*}
-\Psi y_{1n}^m - y_{4n}^m - \nu_{n-1}y_{11}^m \delta_{0k} \delta_{1m} &= r_{1n}^m, \\
\Psi y_{2n}^m - y_{5n}^m + \nu_{n-1}y_{21}^m \delta_{0k} \delta_{1m} &= r_{2n}^m, \\
-\Psi y_{3n}^m - y_{6n}^m - \nu_{n-1}y_{31}^m \delta_{0k} \delta_{1m} &= r_{3n}^m, \\
\Psi y_{4n}^m - b_{41}y_{1n}^m - b_{43}y_{3n}^m - b_{45}y_{5n}^m + \nu_{n-1}y_{41}^m \delta_{0k} \delta_{1m} &= r_{4n}^m, \\
-\Psi y_{5n}^m - b_{54}y_{4n}^m - b_{56}y_{6n}^m - \nu_{n-1}y_{51}^m \delta_{0k} \delta_{1m} &= r_{5n}^m, \\
\Psi y_{6n}^m - b_{61}y_{1n}^m - b_{63}y_{3n}^m - b_{65}y_{5n}^m + \nu_{n-1}y_{61}^m \delta_{0k} \delta_{1m} &= r_{6n}^m,
\end{align*}
\]

where the series \(r\) represent the known part of the equation, \(\Psi = k \omega + m \nu_0\), as before, and \(r_j\) is of the same type as \(y_j\).

When \((k, m) \neq (0, 1)\) we must solve,

\[
\begin{pmatrix}
-\Psi & 0 & 0 & -1 & 0 & 0 \\
0 & \Psi & 0 & 0 & -1 & 0 \\
0 & 0 & -\Psi & 0 & 0 & -1 \\
-b_{41} & 0 & -b_{43} & \Psi & -b_{45} & 0 \\
0 & 0 & 0 & -b_{54} & -\Psi & -b_{56} \\
-b_{61} & 0 & -b_{63} & 0 & -b_{65} & \Psi
\end{pmatrix}
\begin{pmatrix}
y_{1n}^m \\
y_{2n}^m \\
y_{3n}^m \\
y_{4n}^m \\
y_{5n}^m \\
y_{6n}^m
\end{pmatrix}
= \begin{pmatrix}
r_{1n}^m \\
r_{2n}^m \\
r_{3n}^m \\
r_{4n}^m \\
r_{5n}^m \\
r_{6n}^m
\end{pmatrix}.
\]

The case \((k, m) = (0, 0)\) gives a singular matrix, but since \(r_2, r_4\) and \(r_6\) are sinus type series, we have \(r_{2n}^0 = r_{4n}^0 = r_{6n}^0 = 0\) and so we can choose \(y_{2n}^0 = y_{4n}^0 = y_{6n}^0 = 0\) and to solve for the remaining equations.

Finally for the case \((k, m) = (0, 1)\) (which only appears when \(n\) is odd) the previous matrix is again singular due to the determination of \(\nu_0\) during the computation of the
part of order one. But then we solve,

\[
\begin{pmatrix}
-\nu_0 & 0 & 0 & -1 & 0 & 0 & -y_{11}^0 \\
0 & \nu_0 & 0 & 0 & -1 & 0 & y_{21}^0 \\
0 & 0 & -\nu_0 & 0 & 0 & -1 & -y_{31}^0 \\
-b_{41} & 0 & -b_{43} & \nu_0 & -b_{45} & 0 & y_{41}^0 \\
0 & 0 & 0 & -b_{54} & -\nu_0 & -b_{56} & -y_{51}^0 \\
-b_{61} & 0 & -b_{63} & 0 & -b_{65} & \nu_0 & y_{61}^0 \\
\end{pmatrix}
\begin{pmatrix}
y_{1n}^0 \\
y_{2n}^0 \\
y_{3n}^0 \\
y_{4n}^0 \\
y_{5n}^0 \\
y_{6n}^0 \\
\nu_{n-1} \\
\end{pmatrix}
= 
\begin{pmatrix}
r_{1n}^0 \\
r_{2n}^0 \\
r_{3n}^0 \\
r_{4n}^0 \\
r_{5n}^0 \\
r_{6n}^0 \\
\end{pmatrix}
\]

for \( y_{1n}^0, y_{2n}^0, y_{3n}^0, y_{4n}^0, y_{5n}^0, y_{6n}^0 \) and \( \nu_{n-1} \).

### 6.6 Sample of Orbits and Tests

Using the second algorithm of the previous section for the implementation of the Lindstedt–Poincaré procedure, we have done computations of quasihalo orbits in different situations: \( L_1 \) and \( L_2 \) equilibrium points in the Sun–Earth+Moon and Earth–Moon systems. We present in this section the results obtained for the \( L_1 \) point of the Sun–Earth+Moon system. The ones corresponding to the other equilibrium points or systems are qualitatively similar. All the computations have been done using order 25, both in the expansions in powers of \( \gamma \) and in the order of harmonics of the Fourier developments.

As a sample of the results obtained, figures 14, 15, 16 and 17 show the projections on the coordinate planes of some orbits. These orbits have been drawn using the results of the LP procedure. See later for the difference with respect to real orbits of the RTBP. All of them have been computed taking as backbone halo orbit one with \( z \)-amplitude \( \beta = 0.2 \). The value of the amplitude in km can be obtained multiplying this value by the unit of distance, which is taken equal to the distance from the equilibrium point to the nearest primary. In this case the primary is the Earth+Moon barycentre and the equilibrium point is \( L_1 \), so \( \beta = 0.2 \times 1497610 \text{km} \approx 300 \text{,}000 \text{km} \). The four figures show quasihalo orbits with different \( \gamma \) amplitudes around this halo orbit. The first one, in figure 14, has \( \gamma = 0.032 \approx 48 \text{,}000 \text{km} \) and the motion is clearly quasiperiodic on a small torus around the halo orbit. Figure 15 corresponds to \( \gamma = 0.067 \approx 100 \text{,}340 \text{km} \). For this value of the amplitude, there is an almost resonance between the two basic frequencies of the motion, \( \omega \) and \( \nu \), and so the motion on the torus seems to be periodic (\( \omega = 2.05584478... \), and \( \nu = 0.12105618.., \) so \( \omega/\nu = 16.98.. \)). Figure 16 is similar to figure 14, but with a larger value of \( \gamma \) which is very close to the border of the “practical region of convergence”, as we will see later. Finally, figure 17 shows an orbit having some strange bendings. For this orbit, the value of \( \gamma \) is outside the practical region of convergence. All the orbits have been plotted for a time span corresponding to 40 revolutions of the basic halo orbit; this means 122 adimensional time units or, equivalently, 19.5 years.

To have an idea of the thickness (that is, the size in a direction transversal to the backbone halo orbit) of the computed tori around the halo orbits, we have plotted their intersections with the surface of section \( z = 0, \dot{z} > 0 \), for different \( \gamma \) amplitudes. The results are shown in figure 18. In the left hand side figure, the points filling the different
Figure 14: Projections on the coordinate planes and a 3D representation of the quasihalo orbit with $\beta = 0.2$ and $\gamma = 0.032$. For all the representations we have used the normalized reference system centered at the equilibrium point. In the 3D representation the aspect ratio is not 1:1:1.
Figure 15: Projections on the coordinate planes and a 3D representation of the quasihalo orbit with $\beta = 0.2$ and $\gamma = 0.067$. For this value of $\gamma$ we have almost resonance between the two basic frequencies and so the motion is very close to periodic.
Figure 16: Projections on the coordinate planes and a 3D representation of the quasihalo orbit with $\beta = 0.2$ and $\gamma = 0.082$. This value of $\gamma$ is close to the border of the region of practical convergence of the expansions.
Figure 17: Similar to the previous figures for $\beta = 0.2$ and $\gamma = 0.090$. For this value of $\gamma$ the expansions are out of the range of practical convergence.

Figure 18: Intersections of the quasihalo orbits with $z = 0$, $\dot{z} > 0$. All the orbits have $\beta = 0.2$. In the left figure the different “curves” correspond to $\gamma = 0.02$ (inner curve), 0.03, 0.04, 0.05 (outer curve). In the right figure the values of $\gamma$ (0.035 in the inner curve, 0.051 and 0.067 in the outer curve) are such that the motion on the tori is very close to periodic.
“curves” correspond to the quasihalo orbits with $\gamma = 0.02$, 0.03, 0.04 and 0.05. All the motions look quasiperiodic and the orbits associated with each one of the curves fill up the corresponding torus. The points of the right hand side figure look like isolated points (at least if we represent a not very large number of intersections). For the amplitudes selected for this last figure, there is an (almost) resonance between the halo frequency and the normal to the halo frequency, which is close to 17:1 for the outer points ($\gamma = 0.067$), 20:1 for the intermediate points ($\gamma = 0.051$) and 23:1 for the inner points ($\gamma = 0.035$). The pattern of these two figures is different from the standard one that is obtained when one computes iterates of the Poincaré map around a fixed point of a Hamiltonian system. In this latter case the “invariant curves” do not intersect each other as in our figures. This is due to the fact that in our case we have not kept fixed the value of the energy (Hamiltonian) for the different orbits displayed. We have done this for figure 19. In this figure, the value of the energy, given by the Hamiltonian of the RTBP in normalized units, and assigning the value zero to the $L_1$ point, is $h = 0.6$, which is the one corresponding to the periodic halo orbit with $z$–amplitude $\beta = 0.26097534$ (391 000 km approximately). This orbit should be represented as a fixed point of the Poincaré map located inside all the curves. The other curves correspond to quasihalo orbits of different values of $\gamma$ that have been computed for different values of $\beta$ too, in order to be on a fixed level of the energy.

For reference, a part of the planar Lyapunov orbit on the same level of energy is also shown. This illustrates that “good” results are obtained, at least, up to one half of the distance to the planar Lyapunov orbit, for this energy. This means that up to this value of $\gamma$ we are within the practical radius of convergence of the expansions.

Figure 19: Intersection of the quasihalo orbits with $z = 0$. All the orbits have the same value of the Hamiltonian, $h = 0.6$. The different curves correspond (in the sense of decreasing sizes) to quasihalo orbits with $(\beta, \gamma) = (0.16, 0.0637702)$ (outer curve), $(0.17, 0.0610280)$, $(0.18, 0.0583477)$, $(0.19, 0.0553808)$, $(0.20, 0.0520384)$, $(0.21, 0.0482263)$, $(0.22, 0.0438212)$, $(0.23, 0.0386083)$, $(0.24, 0.0321922)$, $(0.25, 0.0235894)$, $(0.26, 0.0071181)$ (inner curve). For reference, also a part of the planar Lyapunov orbit on the same level of energy is shown (as continuous line).

Concerning the instability properties of the quasihalo orbits, they are essentially the same as those of the base halo orbits: when $\beta$ increases, the orbits become less unstable,
but in all cases they display a strong instability. This fact becomes clear when we compute the maximal Lyapunov number associated with these orbits. As the time $t$ increases, the maximal Lyapunov number stabilizes in a fast way around a positive value. In figure 20 we show a typical behaviour for the case of a quasihalo orbit with $\beta = 0.15$ and $\gamma = 0.02$ of the S–E+M around the $L_1$ point for the first 1000 days. In this case it tends in a rather fast way to a value close to 2.415. This value is always very close to the Lyapunov number associated with the periodic halo orbit. Notice that despite previous plots were done for 122 adimensional time units, this time span is too small to get a good enough determination of the maximal Lyapunov number. In figure 20 a value of 1000 units has been used. This is possible because the analytic approximation of the orbit is known. In spite of this instability, the spacecraft can be controlled in practice by rather cheap station-keeping strategies. The related geometric ideas and the technical details can be found, for the case of halo orbits, in [7], [8], [3], [10] and [9]. The modifications to deal with quasihalo orbits are easy.

![Figure 20: Evolution of the approximation to the maximal Lyapunov number versus time (in adimensional units) for a quasihalo orbit with $\beta = 0.15$ and $\gamma = 0.02$ of the S–E+M around the $L_1$ point.](image)

7 Multiple Shooting Techniques

The purpose of this section is to get solutions of more realistic equations of motion (i.e., Newton’s equations using some JPL model for the motion of the bodies of the solar system) that remain close to the solutions obtained for the RTBP in the previous sections. As a first restriction, considering that we are going to compute these solutions numerically, we will fix a time span, which means we will fix an initial epoch (now the system is non autonomous) and a length for the time span.

The general idea is to use a multiple shooting method similar to the one used for the numerical solution of boundary-value problems (see [12]). This method is useful for the computation of highly unstable periodic orbits with very long periods. Our problem is, in some sense, close to this one. As in the standard procedure, first we split the total time span into a number of shorter subintervals selecting, for instance, $N$ equally spaced points

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\( t_1, t_2, \ldots, t_N \). (\( t_1 \) is the initial epoch and \( t_N - t_1 \) the length of the time interval mentioned above). Different time intervals can be used, but we have chosen here all of them equal. Denote by \( \Delta t = t_{i+1} - t_i \) and by
\[
Q_i = (t_i, x_i, y_i, z_i, \dot{x}_i, \dot{y}_i, \dot{z}_i)^T, \quad i = 1, 2, \ldots, N
\]
the points on a fixed quasi-halo orbit of the RTBP, equally spaced (\( \Delta t \)) in time, computed using the formal expansions. Let \( \phi(Q_i) \) be the image of the point \( Q_i \) under the flow associated to the equations of motion in the solar system after an amount of time \( \Delta t \).

As, in this way, the epochs \( t_i \) are fixed, we can write
\[
Q_i = (x_i, y_i, z_i, \dot{x}_i, \dot{y}_i, \dot{z}_i)^T, \quad i = 1, 2, \ldots, N
\]
and if all the points \( Q_i \) would be on the same orbit of the new equations, then \( \phi(Q_i) = Q_{i+1} \) for \( i = 1, \ldots, N - 1 \). As this is not the case, we need to change the starting values to fulfill the matching conditions. In this way, we must solve a set of \( N - 1 \) nonlinear equations, which can be written as
\[
F \left( \begin{array}{c}
Q_1 \\
Q_2 \\
\vdots \\
Q_N 
\end{array} \right) = \left( \begin{array}{c}
\phi(Q_1) \\
\phi(Q_2) \\
\vdots \\
\phi(Q_{N-1}) 
\end{array} \right) - \left( \begin{array}{c}
Q_2 \\
Q_3 \\
\vdots \\
Q_N
\end{array} \right) = \left( \begin{array}{c}
\phi(Q_1) \\
\phi(Q_2) \\
\vdots \\
\phi(Q_{N-1})
\end{array} \right) - \left( \begin{array}{c}
Q_2 \\
Q_3 \\
\vdots \\
Q_N
\end{array} \right) = 0.
\]

We use Newton’s method to solve the above system. If \( Q^{(j)} = (Q_1^{(j)}, Q_2^{(j)}, \ldots, Q_N^{(j)})^T \), denotes the \( j \)-th iterate of the procedure, Newton’s equations can be written as
\[
DF(Q^{(j)}) \cdot (Q^{(j+1)} - Q^{(j)}) = -F(Q^{(j)}),
\]
where the differential of the function \( F \) has the following structure
\[
DF = \begin{pmatrix}
A_1 & -I \\
A_2 & -I \\
& \ddots & \ddots \\
& & A_{N-1} & -I
\end{pmatrix},
\]
with
\[
D\Phi = \begin{pmatrix}
A_1 \\
A_2 \\
& \ddots \\
& & A_{N-1}
\end{pmatrix}.
\]

As each of the transition matrices, \( A_i \), that appear in \( D\Phi \) are \( 6 \times 6 \), at each step of the method we have to solve a system of \( (N - 1) \times 6 \) equations with \( 6 \times N \) unknowns, so some additional conditions must be added. This is the only difference with the standard multiple shooting method and is due to the fact that our problem is not a real boundary-value one. As additional equations we could fix some initial and final conditions at \( t = t_0 \) and \( t = t_N \). This strategy has been used in [7] and [8] for the numerical refinement of halo orbits. In this case one must take care with the choice because the problem can be
ill conditioned from the numerical point of view. This is because the matrix $DF(Q)$ has a very large condition number. If we assume that $\Delta t$ is of the order of one revolution, all the matrices $A_i$ are similar and if the largest eigenvalue is $\lambda \approx 1500$, the smallest is $\lambda^{-1}$, so the condition number is of the order of $\lambda^2$. To avoid this bad conditioning, we can choose a small value for $\Delta t$. In this case the largest eigenvalue of $A_i$ is not so large, but as the number of points $Q_i$ increases (if we want to cover the same time span) the instability is transferred to the procedure for solving the linear system. Also, the extra boundary conditions can force the solution in a non natural way giving convergence problems when we try to compute the orbit for a long time interval.

To avoid this, we can apply Newton’s method directly. As the system has more unknowns than equations, we have (in general) an hyperplane of solutions. From this set of solutions we try to select the one closer to the initial orbit used to start the procedure. This is done by requiring the correction to be minimum with respect to some norm (i.e. the euclidean norm, see [11]). The use of the normal equations must be avoided because they are usually ill conditioned too. In our situation we have used the structure of the equations in the following way.

Denoting by $\Delta Q^{(j)}$

$$\Delta Q^{(j)} = Q^{(j+1)} - Q^{(j)},$$

if we require $\|\Delta Q^{(j)}\|_2$ to be minimum, using the Lagrange function $L(\Delta Q, \mu)$ with (vector) multiplier $\mu$

$$L(\Delta Q, \mu) = \Delta Q^T \cdot \Delta Q + \mu^T \cdot (F(Q) + DF(Q) \cdot \Delta Q),$$

we get

$$\Delta Q^{(j)} = -DF(Q^{(j)})^T \cdot [DF(Q^{(j)}) \cdot DF(Q^{(j)})^T]^{-1} \cdot F(Q^{(j)}),$$

which gives the value of $\Delta Q^{(j)}$ explicitly. Let $M = DF(Q^{(j)}) \cdot DF(Q^{(j)})^T$. This is a symmetric band matrix with the following pattern

$$M = \begin{pmatrix}
I + A_1A_1^T & -A_1 & & & \\
-A_1^T & I + A_2A_2^T & -A_2 & & \\
& \ddots & \ddots & \ddots & \\
& & -A_{N-3}^T & I + A_{N-2}A_{N-2}^T & -A_{N-2} \\
& & & -A_{N-2}^T & I + A_{N-1}A_{N-1}^T
\end{pmatrix}.$$  

We introduce additional variables, $Z^{(j)}$, by $M^{-1}F(Q^{(j)}) = Z^{(j)}$, and then equation (20) becomes

$$M \cdot Z^{(j)} = F(Q^{(j)}),$$

$$\Delta Q^{(j)} = -DF(Q^{(j)})^T \cdot Z^{(j)}.$$  

(21)  

(22)
Now we use a block Cholesky factorization to express \( M \) as

\[
\begin{pmatrix}
I & L_2 & I \\
& \ddots & \ddots \\
L_{N-1} & I & I
\end{pmatrix}
\begin{pmatrix}
D_1 & & \\
& \ddots & \\
& & D_{N-1}
\end{pmatrix}
\begin{pmatrix}
I & L_2^T \\
& \ddots & \\
& & I & L_{N-1}^T
\end{pmatrix},
\]

obtaining the following recursive relations

\[
\begin{align*}
D_1 &= I + A_1 A_1^T, \\
L_i &= -A_i D_i^{-1}, & i &= 2, 3, ..., N-1, \\
D_i &= I + A_i A_i^T - L_i D_{i-1} L_i^T, & i &= 2, 3, ..., N-1.
\end{align*}
\]

With this factorization, system (21) can be solved recursively using intermediate variables, \( X, Y \), with block components \( X_1, ..., X_{N-1}, Y_1, ..., Y_{N-1} \) such that \( Y = Z^{(j)} \) and \( F = (F_1, ..., F_{N-1}) \)

\[
\begin{align*}
X_1 &= F_1(Q^{(j)}), \\
X_k &= F_k(Q^{(j)}) - L_k X_{k-1}, & k &= 2, 3, ..., N-1, \\
Y_{N-1} &= D_{N-1}^{-1} X_{N-1}, \\
Y_k &= D_k^{-1} X_k - L_{k+1}^T Y_{k+1}, & k &= N-2, N-3, ..., 1.
\end{align*}
\]

The value of \( \Delta Q^{(j)} \) can be computed using (22). From these last equations it is clear that the "dangerous" matrices are the \( D_k \) that appear in the recursive computation of \( X_k \) and \( Y_k \). But now these matrices, if \( \Delta t \) is small, are close to diagonal matrices with all the diagonal elements close to \(-1/2, -2/5, -5/12, \ldots\) for \( k = 2, 3, 4, \ldots\) respectively, so that they do not amplify the errors at each step of the procedure. Even if \( \Delta t \) is close to the time required for one revolution, the spectral radius of these matrices is close to one.

### 7.1 Some numerical examples

We have used the above results to produce orbits in the model of solar system given by the JPL ephemeris DE403. In order to illustrate the procedure we will first show the details of some iterations of the computation of a particular solution.

For this first example, to start the algorithm we use as initial nodes, \( Q_i \), that is, the components of \( Q^{(0)} \), points on a quasisolated orbit of the Sun–Earth+Moon system around the \( L_1 \) point with \( \beta = 0.20 \) and \( \gamma = 0.08 \). Although these values are slightly beyond the region of practical convergence the shooting method is still able to cope with this difficulty (see [6]). We fix the initial epoch to be January 1 of the year 2000, and we use 40 nodes with a time step, \( \Delta t \), between two consecutive ones of 180 days. This covers a total time span of 19.7 years. So, the total number of revolutions “around” the equilibrium point \( L_1 \) is approximately 39 and we have taken 1 point (approximately) on each revolution to perform the multiple shooting. In figure 21 we show the \((x, y)\) projection of the orbit after different iterations of the procedure. All the figures are represented in normalized coordinates centered at the \( L_1 \) point. The first plot corresponds
Figure 21: \((x, y)\) projections of the orbits obtained with the multiple shooting procedure at different steps. The figure on the left upper corner is the orbit of the RTBP, computed with the expansions, from which the initial points \(Q_i\) are taken. The orbits with large jumps discontinuities are the ones obtained after the first two iterations. The figure on the right lower corner is the orbit computed after 8 iterations. The initial orbit is a quasihalo orbit with \(\beta = 0.2\) and \(\gamma = 0.08\).
to the orbit, computed with the analytical expansions, from where the points $Q_i$ were taken. It is an approximate solution (due to the truncation and asymptotic character of the series) of the RTBP equations of motion. The next two, showing large discontinuities at some points, are the results obtained after the first two iterations. The different pieces that constitute the orbit do not match at the nodes in these first steps because the initial conditions were taken from a solution of the RTBP and now we are integrating these initial conditions in a model including all the bodies of the solar system with its real motion. These discontinuities are so large because of the highly unstable character of the solution. The last plot corresponds to the orbit computed after 8 iterations. The discontinuities that appear in the first iterations are reduced to “zero” by the method.

In the first step, adding the corrections applied at all the nodes, the total correction in position ($|\Delta Q_{1,2,3}^{(0)}|^2 + |\Delta Q_{7,8,9}^{(0)}|^2 + \ldots$) is of 319600.6 km and of 9360.6 km/day in velocity, which means an average value for the corrections at each point of 8000 km and 235 km/day. After eight iterations the total amount of the corrections has been reduced to 37 mm and less than 1 mm/day, for positions and velocities, respectively. Taking shorter time intervals between consecutive nodes, the norm of the function $F$ is much smaller at the first steps and the number of Newton iterations decreases. For the Sun–Earth+Moon system, a value of $\Delta t$ equal to 7 days requires no more than 4 or 5 iterations to get a final solution with discontinuities at the nodes smaller than tracking errors. For the Earth–Moon system, the computations must be done more carefully and a time step of 1/2 day has given good results.

<table>
<thead>
<tr>
<th>#</th>
<th>Position (Km)</th>
<th>Velocity (Km/day)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1173580E+07</td>
<td>0.5122975E+05</td>
</tr>
<tr>
<td>2</td>
<td>0.1342529E+06</td>
<td>0.6721532E+04</td>
</tr>
<tr>
<td>3</td>
<td>0.9088254E+05</td>
<td>0.4065216E+04</td>
</tr>
<tr>
<td>4</td>
<td>0.8446439E+03</td>
<td>0.4352568E+02</td>
</tr>
<tr>
<td>5</td>
<td>0.5525061E+02</td>
<td>0.2369489E+01</td>
</tr>
<tr>
<td>6</td>
<td>0.3735932E−03</td>
<td>0.2989727E−04</td>
</tr>
<tr>
<td>7</td>
<td>0.1056651E−04</td>
<td>0.5026301E−06</td>
</tr>
<tr>
<td>8</td>
<td>0.3753297E−05</td>
<td>0.1723968E−06</td>
</tr>
</tbody>
</table>

Table 5: Euclidean norm of the function $F$ evaluated at the nodes at the first 8 steps of the multiple shooting. In the second column the norm of the first three components of $F$ is given (error in position) and in the second for the last three ones (error in velocity).

Table 5 gives the values of the function $\|F(Q_{1}^{(j)},Q_{2}^{(j)},...,Q_{40}^{(j)})\|_2$ at the different steps of the algorithm. We have separated the components in position from the ones in velocity. After the iteration number 8 no further improvement is achieved.

We have done many refinements of quasihalo orbits in all the situations. For the Sun–Earth+Moon system the refinement can be done without any problem for all kinds of orbits and for very large time spans (almost all the refined orbits that we have computed cover a time interval of 20 years). In the Earth–Moon system the situation is not so nice. The refinements can also be done without any problem for a time interval of 6–8 years. For larger intervals the procedure must be modified slightly. This is done by requiring to
the norm of the correction, $\|\Delta Q^{(j)}\|_2$, to be minimum for a reduced set of nodal points and at the other nodes computing the correction without any constrain (see [1] for the details and [2] for an illustration). Also the time step between two consecutive nodes must be shorter, between 1/2 and 1 day to get a satisfactory solution with few iterates.

As a test for the robustness of the method we have done the following experiment. As one can see in [6], for a given value of the $z$-amplitude $\beta$, there is a maximum amplitude, $\gamma$, of the quasiperiodic orbits that can be reached with the analytical expansions computed. Some of these invariant tori, that almost foliate the phase space when we compute them up to a given order, do not exist for the RTBP due to the resonances that destroy them (or they exist, as suggested by computation on the center manifold, but the LP procedure is unable to provide good approximations). On the other hand it may happen that for these more realistic equations of motion, there exist these kind of quasihalo orbits with an amplitude larger than the maximum one obtained using the asymptotic expansions.

For this maximum $\gamma$ amplitude we have done two estimates: one comparing the expansions with the direct numerical computation and the other estimating the residual acceleration; here we have taken the most pessimistic of both estimates as practical convergence radii. That is, the minimum of the values using the value $10^{-6}$ for the threshold. Then we have used values of $\gamma$ larger than this maximum to compute the nodes, $Q_i$, as initial points to start the multiple shooting procedure. For all the situations explored we have been able to go beyond the maximum value of $\gamma$ given by the radius of convergence, and to obtain solutions of the real system which look like quasiperiodic halo orbits. In table 6 we give, for different values of $\beta$, the values of the practical convergence radii and the maximum value of $\gamma$ reached (using a rough step of 0.005 in $\gamma$), both for the $L_1$ and $L_2$ points of the Sun–Earth+Moon system. All the orbits were computed for a total time span of 7000 days (approximately 19 years) starting at January 1 of the year 2000. Similar results, but for the Earth–Moon system are given in table 7, where the time span has been taken of 2000 days.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>Practical convergence radii</th>
<th>Maximum $\gamma$</th>
<th>Practical convergence radii</th>
<th>Maximum $\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.015</td>
<td>0.035</td>
<td>0.023</td>
<td>0.050</td>
</tr>
<tr>
<td>0.10</td>
<td>0.029</td>
<td>0.065</td>
<td>0.043</td>
<td>0.085</td>
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<tr>
<td>0.15</td>
<td>0.043</td>
<td>0.090</td>
<td>0.061</td>
<td>0.125</td>
</tr>
<tr>
<td>0.20</td>
<td>0.056</td>
<td>0.105</td>
<td>0.079</td>
<td>0.165</td>
</tr>
<tr>
<td>0.25</td>
<td>0.068</td>
<td>0.145</td>
<td>0.093</td>
<td>0.200</td>
</tr>
</tbody>
</table>

Table 6: For different values of $\beta$, values of the minimum practical radii of convergence estimated (see section 6) of the quasihalo orbits expansions and the maximum amplitude reached in the refinement of these orbits. The results correspond to the Sun–Earth+Moon system.

As a sample of the results obtained, in figures 22, 23, 24 and 25 we show some of these orbits for the maximum values of $\gamma$ reached for the four different situations explored. In each of the figures on the left hand side we plot the “pseudo-orbits”, computed with the
Figure 22: Refinements of quasihalo orbits with $\gamma$ amplitudes out of the region of convergence of the expansions. In the left figures we plot the quasiperiodic formal “solution” on which the initial points are taken. In the right figures the refined orbits have been represented. From top to bottom, the orbits correspond to values of $\beta$ and $\gamma$ equal to (0.05, 0.035), (0.15, 0.090) and (0.25, 0.145) respectively. All the orbits are in the Sun–Earth+Moon system around the $L_1$ point. Note that the initial and final arcs of the numerically computed orbit can go away from a motion looking like quasiperiodic.
Figure 23: Same as figure 22 but for the $L_2$ point. From top to bottom, the orbits correspond to values of $\beta$ and $\gamma$ equal to (0.05, 0.050), (0.15, 0.125) and (0.25, 0.200) respectively.
Figure 24: Same as figure 22 but for the Earth–Moon system $L_1$ point. From top to bottom, the orbits correspond to values of $\beta$ and $\gamma$ equal to $(0.05, 0.025)$, $(0.15, 0.060)$ and $(0.25, 0.080)$ respectively.
Figure 25: Same as figure 22 but for the Earth–Moon system $L_2$ point. From top to bottom, the orbits correspond to values of $\beta$ and $\gamma$ equal to $(0.05, 0.065)$, $(0.15, 0.075)$ and $(0.25, 0.050)$ respectively.
Note that, due to weak conditions imposed in the multiple shooting procedure, the initial and final arcs of the displayed orbits can go away from a motion looking like quasiperiodic. Indeed, no conditions are imposed on the end points. So, in general, most of the computed orbits are very close to an unstable torus (assuming it exists!) except for the initial arc, close to its stable manifold, and the final arc, close to the unstable one. The large value of the Lyapunov number ensures that these two arcs can be visible for, at most, one “revolution”.

Another point worth to be mentioned is the exceptional behaviour of the quasihalo orbits in the $L_2$ Earth–Moon case when the full solar system is considered. Looking to figures 22, 23 and 24, the tori become thicker from top to bottom, in correspondence with an increase of the amplitudes. The situation is different in figure 25, where the thickest torus appears in the middle plot. Of course, both amplitudes $\beta$ and $\gamma$ play a role, but there is also a strong effect due to the 2:1 resonance between the frequency of the halo orbit and the synodical frequency of the Sun in the Earth–Moon system (see [1] for details). This do not happens in the $L_1$ Earth–Moon case.

### 8 Computation of Invariant Manifolds

#### 8.1 Numerical Approximations of Invariant Manifolds

We start with the simplest example. The case of invariant manifolds of a fixed point of a diffeomorphism.

Assume that $p = (x^*, y^*)$ is a fixed point of

$$ P : U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2, $$

with characteristic exponents $\lambda > 1$ and $\mu \in (0, 1)$. We want to approximate $W^s(p)$ (for $W^u(p)$ just use $P^{-1}$).

Let $(x(t), y(t))$ be a parametric representation of the manifold around $p$, with $p = (x(0), y(0))$ and

$$ P(x(t), y(t)) \approx (x(\lambda t), y(\lambda t)). $$

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>Practical convergence radii</th>
<th>Maximum $\gamma$</th>
<th>Practical convergence radii</th>
<th>Maximum $\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.015</td>
<td>0.025</td>
<td>0.044</td>
<td>0.060</td>
</tr>
<tr>
<td>0.10</td>
<td>0.029</td>
<td>0.045</td>
<td>0.081</td>
<td>0.085</td>
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<td>0.042</td>
<td>0.060</td>
<td>0.075</td>
<td>0.075</td>
</tr>
<tr>
<td>0.20</td>
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<td>0.065</td>
<td>0.052</td>
<td>0.065</td>
</tr>
<tr>
<td>0.25</td>
<td>0.069</td>
<td>0.080</td>
<td>0.033</td>
<td>0.050</td>
</tr>
</tbody>
</table>

Table 7: Same as table 6 but for the Earth–Moon system.
This can be as simple as the linear approximation,

\[ (x(t), y(t)) = (x^* + tv_1, y^* + tv_2), \]

where \( v = (v_1, v_2) \) is a normalized eigenvector of the eigenvalue \( \lambda \) or any other higher order approximation.

First we check up to which value of \( t \) the difference

\[ \| P(x(t), y(t)) - (x(\lambda t), y(\lambda t)) \| \]

is less than some given tolerance. Let us assume \( t_0 \) be this value. In general there is no need in being extremely restrictive concerning the tolerance because the small errors tend to decrease (at least locally) due to the compression along the stable direction.

In the next step we take a fundamental domain generated by \( t \) ranging in \((t_0/\lambda, t_0)\).

Finally, we have a simple method which generates a big piece of \( W^u(p) \) by taking \( n \) points in this domain and performing iterates of these points.

However, taking the points and proceeding in this way, one has in general in the iterates too many points closer to \( p \) and too few far away. So some care must be taken in the choice. Also some care must be taken to prevent from large bendings of the manifold. This can be achieved choosing adequately the step, \( \Delta t \), on the fundamental domain (which obviously can not be kept fix on the full domain) and performing a given number of iterations, say \( k \), with this step. When the value of \( t \) is greater than \( t_0 \), then both the parameter \( t \) and the step, \( \Delta t \), must be divided by the eigenvalue and \( k \) can be replaced by \( k + 1 \).

This same method can be applied to obtain unstable invariant manifolds of periodic orbits with one unstable direction and using Poincaré sections.

A more difficult question is how to obtain higher dimensional invariant manifolds. The main difficulty is due to the fact that if we have \( k \) unstable directions (with related eigenvectors \( v_j, j = 1, \ldots, k \) and eigenvalues \( 9\lambda_j, j = 1, \ldots, k, |\lambda_j| > 1 \)), starting the iterations from a point near \( p \) of the form (using linear approximation),

\[ p + \sum_{j=1}^{k} t_j v_j, \]

with \( \sum_{j=1}^{k} t_j^2 \) small, we escape essentially in the direction of the dominant eigenvalue. This is, if \( |\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \ldots \) for almost any \( k \)-ple of values \((t_1, \ldots, t_k)\) we obtain points very close to the ones generated by iteration and starting at \( p + t_1 v_1 \). A general rule to overcome this behaviour consists on the use of high order expansions. High order expansions allow us to start the iterates at a relatively big distance of \( p \).

### 8.2 Local Approximation of the Stable Manifold of a Quasi-Halo Orbit

We are going to see how to compute the linear approximation of the stable manifold of a quasiperiodic halo orbit (qpo).
We know that if an orbit is periodic, the eigenvectors of the variational matrix computed over one period of the orbit (i.e., the monodromy matrix) give the directions of the tangent spaces to the invariant manifolds at the initial point.

Now, our orbits are not periodic but quasiperiodic, although with small deviations from the halo periodic orbit of the restricted three body problem. Then, the eigenvector associated with the eigenvalue with smallest absolute value of the variational matrix, computed over a revolution, approximates quite well the stable direction when the orbit crosses \( y = 0 \).

We want to use the stable manifold of quasiperiodic orbits computed for a given time span, covering several revolutions. If we want to use this manifold for the transfer, we shall be interested in the stable manifold associated to the initial revolutions.

Due to possible small bends of the manifold because of the boundary conditions, adopted in the computations of the qpo by a parallel shooting procedure, the first revolution is skipped and we compute the stable manifold from the beginning of the second one, which means to start the computation of the stable manifold in the third cut of the orbit with \( y = 0 \). From this point we compute the stable direction in a set of points of the halo orbit usually equally spaced on the whole revolution.

Instead of computing the stable direction associated to a selected revolution, and in order to avoid the small discontinuity obtained in the stable direction when we go from one revolution to the next one, the variational matrix \( \bar{A} \) associated to the whole quasiperiodic orbit has been taken as a monodromy matrix.

We recall that \( \bar{A} \) is a \( 7 \times 7 \) matrix because the time was added in the equations of motion in order to get an autonomous system and its pattern is:

\[
\bar{A} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
\bar{g}_1 & 0 & 0 & 0 & 0 & 0 \\
\bar{g}_2 & \bar{g}_3 & \bar{A} & 0 & 0 & 0 \\
\bar{g}_4 & \bar{g}_5 & 0 & \bar{A} & 0 & 0 \\
\bar{g}_6 & 0 & \bar{g}_4 & \bar{g}_5 & \bar{A} & 0 \\
\end{pmatrix},
\]

where \( \bar{A} \) is the part of \( \bar{A} \) associated to positions and velocities. We note that if \( v \) is an eigenvector of \( A \), then \((0, v)\) is an eigenvector of \( \bar{A} \). We shall compute the stable direction for the \( 6 \times 6 \) matrix \( A \).

If \( A_i \) is the \( 6 \times 6 \) matrix related to positions and velocities of a certain variational matrix \( \bar{A}_i \) associated to the \( i \)-th revolution of a numerical qpo made of \( N \) revolutions, we have:

\[ A = A_N \times A_{N-1} \times \ldots \times A_1. \]

Because of possible overflows and rounding errors, due to the fact that the eigenvalue of each \( A_i \) associated to the unstable manifold is of the order of 1700, the above product must not be computed. With the power method applied to

\[ A^{-1} = A_1^{-1} \times A_2^{-1} \times \ldots \times A_N^{-1}, \]
we can compute the eigenvector $v_1$ of the matrix $A$ associated to the stable manifold. Then the eigendirections $v_j$ related to the beginning of each $j$-th revolution are computed by means of:

$$v_j = \frac{A_j^{-1}v_{j+1}}{|A_j^{-1}v_{j+1}|}, \quad j = N, \ldots, 2,$$

where $v_{N+1} = v_1$. The “contraction” (eigenvalue in the case of a periodic orbit) corresponding to the stable direction associated to the $j$-th revolution is:

$$\delta_j = |A_j^{-1}v_{j+1}|^{-1}.$$

When we have the direction of the stable manifold at the beginning of a revolution, we can obtain the stable direction at any intermediate point in the revolution transporting this vector by means of the $6 \times 6$ differential matrix of the flow corresponding to the coordinates of position and velocity. That is, if $v_j(t_0) = v_j$ is the initial vector at time $t_0$ we have

$$v_j(t) = A_j(t)v_j(t_0),$$

where $A_j(t) = Id$.

### 8.3 Globalization of the stable manifold

Once the local approximation is available, the next thing to do is the globalization of the stable manifold. This shall be done at the points and directions computed according to the preceding section.

Given a displacement, $D$, in kilometers from the selected point in the qpo, in the right sense of the stable manifold (for instance the one which gives approaches to the Earth if we want to use the manifold for transfer), initial conditions $X_{0 ws}$ in the linear approximation of the manifold are given by means of:

$$X_{0 ws} = X_{qpo} + D \cdot V_{ws},$$

where $X_{qpo}$ is the selected point of the qpo and $V_{ws}$ is the scaled stable direction in the point $X_{qpo}$.

It must be noted that the magnitude $D$ can not be too small, in absolute value, in order to prevent rounding errors and large integration periods of time when globalizing the manifold. However, it can not be too large because the linear approximation is good near the point $X_{qpo}$. Several tests about this value must be done. In spite of the fact that the attractive character of the manifold towards the qpo was significative even for large values of $D$, taking also into account the integration time during the globalization of the manifold, we suggest values of about 200 or 250 km. In our computations we took $D = 200$. This implies that when we say that we reach the qpo we mean that we are at 200 km of the nominal point $X_{qpo}$ in the qpo. We recall that to reach the qpo has no meaning because of the asymptotic behaviour of the manifold towards the orbit.

Once $X_{0 ws}$ is computed, the only thing that we must do only is to integrate, backwards in time, starting at this initial condition, until a local minimum near the Earth is obtained.
if we are interested in the transfer. However, mainly due to collisions with the Moon and to the big size of the manifold, the algorithm must take into account several problems which can appear during the integration.

Concerning the orbits which pass near the Earth, in which we are interested on, the integration is stopped when we reach a distance to the direction of the Earth, less than its radius.

If the orbit does not collide with the Earth, a Newton procedure is used in order to get the minimum distance to the Earth looking for a zero of the function

\[ F(t) = \vec{r}_E(t) \cdot \dot{\vec{r}}_E(t), \]

where \( \vec{r}_E(t) \) is the vector position of the satellite with respect to the Earth at the epoch \( t \). If the minimum is close to the Earth (we took less than 100000 Km in our computations), and during the numerical integration the distance from the orbit to the Moon is less than some amount (100000 km in our computations), another Newton procedure is started in order to get the minimum distance to the Moon.

References

cada i Anàlisi, Universitat de Barcelona, 1998.


Keeping Strategy for Sun–Earth $L_1$ Libration Point Orbits. Journal of the Astronau-


methods in Celestial Mechanics, D. Benest and C. Froeschlé, editors, pp. 285–330,