Statistical theory of interior-exterior transition and collision probabilities

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Abstract

The dynamics of comets and other solar system objects which have a three-body energy close to that of the collinear libration points are known to exhibit a complicated array of behaviors such as transition between the interior and exterior Hill’s regions, temporary capture, and collision. The invariant manifold structures of the collinear libration points for the restricted three-body problem, which exist for a range of energies, provide the framework for understanding these complex dynamical phenomena from a geometric point of view. In particular, the stable and unstable invariant manifold tubes associated to libration point orbits are the phase space structures that provide a conduit for particles traveling to and from the smaller primary body (e.g., Jupiter). Using the structures around libration points, a statistical theory for the probability of interior-exterior transition and the probability of collision with the smaller primary body can be developed. Comparisons with observations of Jupiter family comets are made.

Introduction

Several Jupiter-family comets such as P/Oterma, P/Gehrels 3, and P/Helin-Roman-Crockett make a transition from heliocentric orbits inside the orbit of Jupiter to heliocentric orbits outside the orbit of Jupiter and vice versa (Carusi, Kresák, Pozzi, and Valsecchi [1985] and Koon, Lo, Marsden, and Ross [2001]). During this transition, the comet can be captured temporarily by Jupiter for one to several orbits around Jupiter (Tancredi, Lindgren, and Rickman [1990], and Howell, Marchand, and Lo [2000]). The Tisserand parameters of these objects, termed the quasi-Hildas (hereafter QHs) by Kresák [1979], are slightly in excess of 3. The possible pre-capture orbital history of D/Shoemaker-Levy 9 (henceforth referred to as SL9) also places it within this group (Benner and McKinnon [1995]).

An important feature of the motion of these comets is that during the phase right before and after their encounter with Jupiter, their orbits pass close to the libration points $L_1$ and $L_2$ of the sun-Jupiter system. This has been pointed out by many authors, including Tancredi, Lindgren, and Rickman [1990], Valsecchi [1992], and Belbruno and Marsden [1997]. Hence objects with low velocity relative to these points (i.e., orbits with apoapse near $L_2$ or periapse near $L_1$) are most likely to be captured (Kary and Dones [1996]).

During the short time just before an encounter with Jupiter, the most important orbital perturbations are due to Jupiter alone, as suggested by the passages of comets by $L_1$ and $L_2$. $N$-body effects of Saturn and the other large planets surely play a significant role over significantly longer times, but we
concentrate here on the time right before a comet’s encounter with Jupiter. To simplify the analysis, we use the most rudimentary dynamical model, namely, the circular, planar restricted three-body model (PCR3BP), to determine the basic phase space structure which causes the dynamical behavior of the QH comets. Furthermore, since the PCR3BP is an adequate starting model for many other systems, results can be applied to other phenomena in the solar system, such as the near-Earth asteroid (NEA) problem, wherein one considers the motion of an asteroid on an energy surface in the sun-Earth system where libration point dynamics are important.

Lo and Ross [1997] suggested that studying the $L_1$ and $L_2$ invariant manifold structures would be a good starting point for understanding the capture and transition of these comets. Koon, Lo, Marsden, and Ross [2000] studied the stable and unstable invariant manifolds associated to $L_1$ and $L_2$ periodic orbits. They took the view that these manifolds, which are topologically tubes within an energy surface, are phase space conduits transporting material to and from Jupiter and between the interior and exterior of Jupiter’s orbit.

In the present paper, we wish to extend the results of Koon, Lo, Marsden, and Ross [2000] to obtain statistical results. In particular, we wish to address two basic questions about QHs and NEAs: How likely is a QH collision with Jupiter or a NEA collision with Earth? How likely is a P/Oterma-like interior-exterior resonance transition? With this work, we put SL9, NEA impacts, and interior-exterior transitions into the broader context of generic motion in the restricted three-body problem.

The paper is broken up into two sections. In section 1, we discuss some phenomena of the QH comets, namely interior-exterior and collisions with Jupiter. In section 2, we frame the above questions as a transport problem, viewing the PCR3BP as the underlying dynamical system. We also summarize the results and suggest future directions.

1 The Quasi-Hilda Group of Comets

The QH group of comets is a small group of strongly Jupiter-interacting comets having a Tisserand parameter slightly above 3, characterized by repeated and long-lasting temporary captures (Benner and McKinnon [1995]). As authors have noted, the capture process frequently moves bodies from orbits outside Jupiter’s orbit to inside Jupiter’s orbit, passing by $L_1$ and $L_2$ in the process of approaching or departing from Jupiter’s vicinity (e.g., Kary and Dones [1996]). We will refer to this type of transition as an interior-exterior transition.

**Interior-Exterior Transition.** In Figure 1(a), we show the interior-exterior transition of QH P/Oterma in a sun-centered inertial frame. The interior orbit is in an exact 3:2 mean motion resonance with Jupiter* while the exterior orbit is near the 2:3 resonance with Jupiter. In Figure 1(b), we show a homoclinic-heteroclinic chain of orbits in the PCR3BP as seen in the rotating frame. This is a set of orbits on the intersection of $L_1$ and $L_2$ stable and unstable manifolds with energies equal to that of P/Oterma. The homoclinic-heteroclinic chain is believed to form the backbone for temporary capture and interior-exterior

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*By *exact*, we mean that P/Oterma orbits the sun three times while Jupiter orbits the sun twice, as seen in an inertial frame.
transition of QHs, as can be seen when the orbit of P/Oterma in the rotating frame is overlayed as in Figure 1(c) (Koon, Lo, Marsden, and Ross [2000]).

Collision with Jupiter. At the time of its discovery, SL9 was only 0.3 AU from Jupiter and broken up into several fragments due to tidal disruption on an earlier approach within the planet’s Roche limit (Marsden [1993]). Integrations indicated that it would collide with the planet (Chodas and Yeomans [1993]), which it subsequently did in July 1994.

Likely Pre-Collision Heliocentric Orbit of SL9. Pre-collision integrations of individual SL9 fragments (Benner and McKinnon [1995]) suggest that the SL9 progenitor approached Jupiter by passing by \( L_1 \) or \( L_2 \) from a short-period heliocentric orbit between either Jupiter and Mars or between Jupiter and Saturn (Figure 2(a)). The distribution of heliocentric \( a \) and \( e \) determined from these fragment integrations are shown in Figure 2(b). The pre-collision fragments have Tisserand parameters of about \( T = 3.02 \pm 0.01 \). From this value and the similarity of the pre-collision orbits to the known QHs, Benner and McKinnon suggest a QH origin for SL9.

Twice as many fragments came from the outer asteroid belt as compared to the inner transjovian region. However, Benner and McKinnon [1995] do not conclude that SL9 originated from the outer asteroid belt. Instead, they say that “the chaos in SL9’s orbit is so strong...that what is being seen is a statistical scrambling of all possible trajectories for an object as loosely bound as SL9.” The bias toward an asteroid origin is a measure of the relative ease of capture (or escape) toward \( L_1 \) versus \( L_2 \), a known result (Heppenheimer and Porco [1977]). The statistical likelihood of a pre-collision interior orbit depends on the relative populations of interacting comets interior or exterior to Jupiter. If there are roughly equal populations, then a pre-collision interior origin is favored.
Figure 2: (a) A typical SL9 trajectory showing the passage past a libration point and subsequent capture. The sun is to the right. (Reproduced from Benner and McKinnon [1995]. According to their terminology, their $L_2$ is our $L_1$, and vice versa.) (b) Heliocentric $a$ and $e$ of possible SL9 progenitor orbits, based on fragment integrations. The positions of selected comets and two major outer belt asteroid groups, the Trojans and the Hildas, are shown. The dashed curves are for Tisserand parameter $T = 3$ (for zero inclination); orbits above the upper curve and below the lower curve have $T > 3$ and are generally not Jupiter-crossing, while those between the two curves ($T < 3$) are Jupiter-crossing. (Reproduced from Benner and McKinnon [1995].)

2 Transport in the Planar Circular Restricted Three-Body Problem

When the dynamics are chaotic, statistical methods may be appropriate (Wiggins [1992]). By following ensembles of phase space trajectories, we can determine transition probabilities concerning how likely particles are to move from one region to another.

Following Wiggins [1992], suppose we study the motion on a manifold $M$. Further, suppose $M$ is partitioned into disjoint regions

$$ R_i, i = 1, \ldots, N_R, $$

such that

$$ M = \bigcup_{i=1}^{N_R} R_i. $$

At $t = 0$, region $R_i$ is uniformly covered with species $S_i$. Thus, species type of a point indicates the region in which it was located initially.

The statement of the transport problem is then as follows:

*Describe the distribution of species $S_i, i = 1, \ldots, N_R$, throughout the regions $R_j, j = 1, \ldots, N_R$, for any time $t > 0$.*

Some quantities we would like to compute are: $T_{i,j}(t)$, the amount of species $S_i$ contained in region $R_j$, and $F_{i,j}(t) = \frac{dT_{i,j}}{dt}(t)$, the flux of species $S_i$ into region $R_j$ (see Figure 3). For some problems, the probability of transport between two regions or the probability of an event occurring (e.g., collision), may be more relevant.
Planar Circular Restricted Three-Body Problem. Here we only review the material concerning the PCR3BP which has relevance toward our discussion of transport. See details in Szebehely [1967] and Koon, Lo, Marsden, and Ross [2001].

Consider motion in the standard rotating coordinate system as shown in Figure 4 with the origin at the center of mass, and the sun and Jupiter fixed on the $x$-axis at the points $(-\mu, 0)$ and $(1 - \mu, 0)$ respectively. Let $(x, y)$ be the position of the comet in the plane, then the equations of motion in this rotating frame are:

\[
\begin{align*}
\ddot{x} - 2\dot{y} &= -U_x^{\text{eff}}, \\
\dot{y} + 2\dot{x} &= -U_y^{\text{eff}},
\end{align*}
\]

where

\[
U^{\text{eff}} = -\frac{1}{2}(x^2 + y^2) - \frac{1 - \mu}{r_1} - \frac{\mu}{r_2}
\]

is the effective potential and the subscripts denote its partial derivatives and $r_1, r_2$ are the distances from the comet to the sun and the Jupiter respectively.

These equations are autonomous and can be put into Hamiltonian form. They have an energy integral:

\[
E = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + U^{\text{eff}}(x, y).
\]

which is related to the Jacobi integral $C$ by $C = -2E$. The Jacobi integral can be expressed approximately in terms of the comet’s semimajor axis, $a$, and eccentricity, $e$, in a form known as the Tisserand parameter, $T$, i.e., $C = T + \mathcal{O}(\mu)$, where

\[
T = \frac{1}{a} + 2\sqrt{a(1 - e^2)}.
\]

The energy manifolds,

\[
\mathcal{M}(\mu, \varepsilon) = \{(x, y, \dot{x}, \dot{y}) \mid E(x, y, \dot{x}, \dot{y}) = \varepsilon\}
\]

where $\varepsilon$ is a constant are 3-dimensional surfaces foliating the 4-dimensional phase space. The Hill’s regions are the projection of the energy manifold onto
the position space is the region in the $xy$-plane where the comet is energetically permitted to move,

$$M(\mu, \varepsilon) = \{(x, y) \mid U^{\text{eff}}(x, y) \leq \varepsilon\}.$$  

The forbidden region is the region which is not accessible for the given energy. See Figure 4(b).

![Diagram showing the libration points and energy regions](image)

**Figure 4:** (a) The rotating frame showing the libration points, in particular $L_1$ and $L_2$, of the planar, circular restricted three-body problem. (b) Energetically forbidden region is gray "C". The Hill’s region, $M(\mu, \varepsilon)$ (region in white), contains a bottleneck about $L_1$ and $L_2$. (c) The flow in the region near $L_2$, showing a periodic orbit around $L_2$ (labeled PO), a typical asymptotic orbit winding onto the periodic orbit (A), two transit orbits (T) and two non-transit orbits (NT). A similar figure holds for the region around $L_1$.

Eigenvalues of the linearized equations at $L_1$ and $L_2$ have one real and one imaginary pair, having a saddle $\times$ center structure. Our main concern is the behavior of orbits whose energy is just above that of $L_2$, for which the Hill’s region is a connected region with an *interior* region (inside Jupiter’s orbit), *exterior* region (outside Jupiter’s orbit), and a *Jupiter* region (bubble surrounding Jupiter). We will use the terminology interior, exterior, and Jupiter regions to mean regions in the Hill’s region and the corresponding regions of the energy surface, $M(\mu, \varepsilon)$. Thus, we have a useful partition for our problem for which we can compute transport properties. These regions are connected by bottlenecks about $L_1$ and $L_2$ and the comet can pass between the regions only through these bottlenecks. Inside each bottleneck, adjacent regions of the (e.g., the interior and Jupiter regions) share a common boundary in the energy surface. This common boundary is known as the transition state and has been used previously in astrodynamical transport calculations (Jaffé, Ross, Lo, Marsden, Farrelly, and Uzer [2002]). For our analysis of transport, we must focus on the bottlenecks.

In each bottleneck (one around $L_1$ and one around $L_2$), there exist 4 types of orbits, as given in Conley [1968] and illustrated in Figure 4(c): (1) an unstable periodic Lyapunov orbit; (2) four cylinders of asymptotic orbits that wind onto or off this period orbit, which form pieces of stable and unstable manifolds; (3) transit orbits which the comet must use to make a transition from one region to the other; and (4) nontransit orbits where the comet bounces back to its original region.

McGehee [1969] was the first to observe that the asymptotic orbits are pieces of the 2-dimensional stable and unstable invariant manifold tubes associated to the Lyapunov orbit and that they form the boundary between
transit and nontransit orbits. The transit orbits, passing from one region to another, are those inside the cylindrical manifold tube. The nontransit orbits, which bounce back to their region of origin, are those outside the tube. Most importantly, to transit from outside Jupiter’s orbit to inside (or vice versa), or get temporarily captured, a comet must be inside a tube of transit orbits, as in Figures 5(a) and 5(b). The invariant manifold tubes are global objects—they extend far beyond the vicinity of the bottleneck, partitioning the energy manifold.

Numerical Computation of Invariant Manifolds. Key to our analysis is the computation of the invariant manifolds of Lyapunov orbits, thus we include some notes on computation methods. Periodic Lyapunov orbits can be computed using a high order analytic expansion (see Llibre, Martinez, and Simó [1985]) or by using continuation methods (Doedel, Paffenroth, Keller, Dichmann, Galan, and Vanderbauwhede [2002]). Their stable and unstable manifolds can be approximated as given in Parker and Chua [1989]. The basic idea is to linearize the equations of motion about the periodic orbit and then use the monodromy matrix provided by Floquet theory to generate a linear approximation of the stable manifold associated with the periodic orbit. The linear approximation, in the form of a state vector, is numerically integrated in the nonlinear equations of motion to produce the approximation of the stable manifold. All numerical integrations were performed with a standard seventh-eighth order Runge-Kutta method.

Interior-Exterior Transition Mechanism. The heart of the transition mechanism from outside to inside Jupiter’s orbit (or vice versa) is the intersection of tubes containing transit orbits. We can see the intersection clearly on a 2-dimensional Poincaré surface-of-section in the 3-dimensional energy manifold. We take our surface to be \( \Sigma(\mu, \epsilon) = \{(y, \dot{y})|x = 1 - \mu, \dot{x} < 0\} \), along a vertical line passing through Jupiter’s center as in Figure 6(a). In Figure 6(b), we plot \( \dot{y} \) versus \( y \) along this line, we see that the tube cross-sections are
distorted circles. Upon magnification in Figure 6(c), it is clear that the tubes indeed intersect.

Figure 6: (a) We take a Poincaré surface-of-section \( \Sigma(\mu, \varepsilon) = \{(y, \dot{y}) | x = 1 - \mu, \dot{x} < 0\} \), along a vertical line through the center of Jupiter \((J)\). Both the \( L_1 \) and \( L_2 \) periodic orbit invariant manifold tubes intersect \( \Sigma(\mu, \varepsilon) \) transversally. (b) On \( \Sigma(\mu, \varepsilon) \), we see the first unstable tube cut for \( L_2 \) and first stable tube cut for \( L_1 \). (c) A small portion of the interior of the tubes intersect—this set in the energy manifold \( \mathcal{M}(\mu, \varepsilon) \) containing the comet orbits which pass from the exterior to the interior region.

Any point within the region bounded by the curve corresponding to the stable tube cut is on an orbit that will go from the Jupiter region into the interior region. Similarly, a point within the unstable tube cut is on an orbit that came from the exterior region into the Jupiter region. A point inside the region bounded by the intersection of both curves (lightly shaded in Figure 6(c)) is on an orbit that makes the transition from the exterior region to the interior region, via the Jupiter region.

**Interior-Exterior Transition Probability.** Note that since \( p_y = \dot{y} + x \) and \( x \) is constant, the \((y, \dot{y})\) plane is a linear displacement of the canonical plane \((y, p_y)\). Furthermore, the action integral around any closed loop \( \Gamma \) on \( \Sigma(\mu, \varepsilon) \),

\[
S = \oint_{\Gamma} p \cdot dq = \oint_{\Gamma} p_y \ dy,
\]

is simply the area enclosed by \( \Gamma \) on the surface-of-section \( \Sigma(\mu, \varepsilon) \) (Meiss [1992]).

The agreement between a Monte-Carlo simulation and a Markov approximation in an earlier paper (Jaffé, Ross, Lo, Marsden, Farrelly, and Uzer [2002]) suggests that for energies slightly above \( L_1 \) and \( L_2 \), there are components of the energy surface for which the motion is “well mixed” (cf. Meiss [1992]). Thus, the Markov approximation is a good one. Let \( R_1 \) be the interior region and \( R_2 \) be the exterior region. In the Markov approximation, the probability of a particle going from region \( R_i \) to \( R_j \) is

\[
P_{ij} = \frac{F_{ij}}{A_j}
\]

where \( A_j \) is the area of the first unstable tube cut on \( \Sigma(\mu, \varepsilon) \), containing transit orbits from \( R_j \) and \( F_{ij} = F_{ji} \) is the area of overlap of the first unstable tube cut from \( R_j \) and the first stable tube cut from \( R_i \) on \( \Sigma(\mu, \varepsilon) \). This transition
probability is exact for one iterate of the Poincaré map; however, it is typically only qualitatively correct for longer times.

In Figure 7, we give the results of the calculations of $P_{12}$ and $P_{21}$ for mass parameter $\mu = 9.537 \times 10^{-4}$ and a variety of energies in the range of QH Jupiter-family comets. This is the probability of a comet to move from the interior to the exterior and vice versa during its first pass through the surface-of-section $\Sigma(\mu, \varepsilon)$.

Figure 7: Interior-exterior transition probabilities for quasi-Hilda Jupiter-family comets. The probability of a comet to move from the interior to the exterior and vice versa during its first pass through the surface-of-section $\Sigma(\mu, \varepsilon)$ is plotted as a function of energy in the planar, circular restricted three-body problem. The energy value of P/Oterma is shown for comparison. Note that interior to exterior transitions are slightly more probable than the reverse transition.

A few comments regarding this result are due. (1) Notice that there is a lower limit in energy, $E_t \approx -1.517$. For $E \leq E_t$, the tube cuts do not overlap and no direct transition is possible. After more loops around Jupiter, transition may be possible (cf. Koon, Lo, Marsden, and Ross [2000]). (2) The probability increases as a function of energy. (3) Quasi-Hilda P/Oterma is located in the region of $\approx 25\%$ probability. (4) Finally, notice that $P_{12} > P_{21}$, which is a result of $A_1 > A_2$, the slight asymmetry we should expect for a mass parameter of this value or larger (cf. Simó and Stuchi [2000]).

Collision Probabilities. Collision probabilities can be computed for objects coming through the $L_1$ and $L_2$ bottlenecks from the interior and exterior regions, respectively. We augment the procedure for computing interior-exterior transition probabilities in the following way. Instead of computing $\mathcal{F}_{ij}$, we now compute the overlap of the first unstable manifold cut with the diameter of the secondary (e.g., Jupiter). Since the surface $\Sigma(\mu, \varepsilon)$ passes through the center of secondary, any particle located on $\Sigma(\mu, \varepsilon)$ with $|y| < R$ will have collided with the secondary, where $R$ is the radius of secondary in units of the
primary-secondary distance. This is illustrated in Figure 8.

![Poincare Section: Tube Intersecting a Planet](image)

Figure 8: The surface-of-section, $\Sigma_{(\mu, \varepsilon)}$, is shown, with $y$ vs. $\dot{y}$. The area inside the first unstable manifold tube cut with $|y| \leq R$ is shown in in black. These are orbits that collide with the surface of the secondary. The two vertical lines are at $y = \pm R$.

There is a singularity at the center of the secondary, $y = 0$ on $\Sigma_{(\mu, \varepsilon)}$, so the calculation is actually performed along a nearby parallel surface-of-section, where $x = 1 - \mu \pm c$, with $c$ a small number on the order of the integration tolerance (the ‘+’ sign is for orbits coming from the exterior, and the ‘−’ for orbits coming from the interior).

Collision probabilities for the sun-Jupiter case ($\mu = 9.537 \times 10^{-4}$, $R = 8.982 \times 10^{-5}$) are given in Figure 9. We notice the following. (1) The probability is not monotonically increasing as in Figure 7. (2) The energy range of possible pre-collision Shoemaker-Levy 9 orbits (from Benner and McKinnon [1995]) lies in the range of highest collision probability, suggesting the utility of this approach. (3) There is an asymmetry in orbits coming from the interior or the exterior, and now there are two lower energy cutoffs, $E_1^c \approx -1.5173$ and $E_2^c \approx -1.5165$, below which no collision can occur on the first pass by Jupiter. The asymmetry may be too slight to differentiate an interior origin from an exterior origin for SL9.

As a final computation, we address the NEA collision problem. For a mass parameter corresponding to the sun-Earth-asteroid problem ($\mu = 3.059 \times 10^{-6}$, $R = 4.258 \times 10^{-5}$), we compute the collision probability. The result is shown in Figure 10. It is interesting that the collision probabilities are nearly twice those for the quasi-Hilda case, even though Jupiter has a much larger mass and radius than the Earth. The asymmetry in interior/exterior originating orbits is not as pronounced as in Figure 9, owing to the smaller value of $\mu$, and $E_1^c \approx E_2^c \approx -1.5 - 4.03 \times 10^{-4}$. 

10
Figure 9: **Collision probabilities for quasi-Hilda comets.** The probability of collision for orbits making their first pass through the surface-of-section $\Sigma_{(\mu,\epsilon)}$ is plotted as a function of energy. The energy range of possible pre-collision D/Shoemaker-Levy 9 orbits is shown for comparison.

**Conclusions**

We address some questions regarding nonlinear comet and asteroid behavior by applying statistical methods to the planar, circular restricted three-body problem. In particular, we make a Markov assumption regarding the phase space and compute probabilities of interior-exterior transition and collision with the secondary. Theory and observation are seen to agree for the comets P/Oterma and D/Shoemaker-Levy 9.

**References**


6. Doedel, E.J., R.C. Paffenroth, H.B. Keller, D.J. Dichmann, J. Galan, and A. Vanderbauwhede [2002], Continuation of periodic solutions in
Figure 10: Collision probabilities for near-Earth asteroids. Note that the collision probabilities are nearly twice those for the quasi-Hilda case in Figure 9, even though Jupiter has a much larger mass and radius than the Earth.


17. Marsden, B.G. [1993], IAU Circ. 5726.


