The Computation of Periodic Solutions of the 3-Body Problem Using the Numerical Continuation Software AUTO

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The Point of This Talk

- Tools such as Swingby, Satellite Tool Kit and LTool are powerful mission design tools, given a good first approximation of a trajectory.
- The mission designer needs a way to generate a good first approximation for an N-body trajectory.
- A tool that can generate a “catalog” of first approximations gives the analyst a wider range of choices for mission design.
- Numerical continuation software such as AUTO2000 is one possible tool for generating a catalog of approximate trajectories.
Outline

• Introduction to the AUTO2000 software package
• Formulation of the Circular Restricted 3-Body Problem for Continuation Methods: The Challenge of Conservative Systems
• Tour of Some Periodic Orbits Emanating from $L_1$
• Orbital Acrobatics and Backflip Orbits
• Mission Applications
• Conclusions
AUTO2000 Overview

• AUTO was originally written in 1980, but was recently redone in C in 2000.

• Computes parameter-dependent families of solutions to nonlinear equations using numerical continuation.

• Used for the bifurcation analysis of algebraic as well as ordinary differential equations (ODEs) after discretization.

• Has been used in many applications (fluid dynamics, cardiac electrophysiology, continuum models of large bio-molecules, flight trim analysis, and the gravitational 3-body problem, to name but a few.)

Basic Equation

• The goal of AUTO2000 is to compute and visualize solution sets of equations of the form

\[ F(x) = 0, \quad F : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n, \]

i.e. an under-determined system where you have one more unknown than equations.

• Away from singularities the solution set of such a system is a 1-dimensional manifold embedded in \((n + 1)\)-dimensional space.

• Many kinds of problems can be phrased in this way, including discretizations of ODE boundary value problems (which includes periodic solutions to the N-body problem).
Point of Numerical Continuation

In some problems, the initial guess $x_{guess}^*$ is not sufficiently accurate for Newton’s method to converge. Instead, embed the problem in a family of problems, where a solution $x_0$ is known for one problem. Then numerically continue, with steps small enough for Newton’s method to converge, until the desired problem is solved.
Pseudo-arclength Continuation

To adapt to the geometry of the solution manifold, we use the arclength as the continuation parameter. The continuation problem is then expressed as

\[ F(x) = 0, \]
\[ (x_1 - x_0)^T \dot{x}_0 = \Delta s . \]
Python interface

- Previous versions of AUTO had a good interface for expert users, but this interface was somewhat difficult for beginners.
- To create a friendlier interface, a wrapper written in the object-oriented language Python was added in AUTO2000. The interface of the flavor of Matlab, but is geared toward continuation methods.
Sample Python Control Script

Just like Matlab, AUTO2000 can be used in two modes. First, it can be used interactively. Second, it is a programming language in which you can express complex computations:

```python

    copydemo('bvp')

    ld('bvp')
    run()  
    sv('bvp')
    ld(s='bvp')
    data = sl('bvp')
    ch("NTST",50)
    for solution in data:
        if solution["Type"] == 'BP':
            ch("IRS", solution["Label"])
            ch("ISW", -1)
            # Compute back
            ch("DS",-pr("DS"))
            run()
            ap('bvp')
    plot('bvp')
```
The Circular Restricted 3-Body Problem (CR3BP)

- Two primary masses move in circular orbits about their barycenter
- Third body with negligible mass influenced by the primaries
- Coordinates in barycentric frame rotating with the primaries
- Parameter $\mu$ is the ratio of the smaller primary mass to the total mass.
- We use $\mu = 0.01215$, for the Earth-Moon system
The Circular Restricted 3-Body Problem (2)

The equations of motion are

$$
\begin{align*}
  x'' &= 2y' + x - (1 - \mu)(x + \mu)r_1^{-3} - \mu(x - 1 + \mu)r_2^{-3}, \\
  y'' &= -2x' + y - (1 - \mu)yr_1^{-3} - \mu yr_2^{-3}, \\
  z'' &= -(1 - \mu)zr_1^{-3} - \mu zr_2^{-3},
\end{align*}
$$

$$
\begin{align*}
  r_1 &= \sqrt{(x + \mu)^2 + y^2 + z^2}, \\
  r_2 &= \sqrt{(x - 1 + \mu)^2 + y^2 + z^2},
\end{align*}
$$

The system has a single integral of motion, the Jacobi integral

$$
C' = (x^2 + y^2) + \frac{2(1 - \mu)}{r_1} + \frac{2\mu}{r_2} - (v_x^2 + v_y^2 + v_z^2)
$$
Computation of Periodic Solutions: Problem Formulation

- To apply numerical continuation methods we first phrase the computation of a periodic orbit as a two-point boundary value problem by adding appropriate boundary conditions

\[ x(1) = x(0), \quad y(1) = y(0), \quad z(1) = z(0) \]
\[ v_x(1) = v_x(0), \quad v_y(1) = v_y(0), \quad v_z(1) = v_z(0) \]

...to impose unit periodicity and solve for the unknown period \( T \) as part of the numerical continuation procedure.

- To solve the two-point boundary value problem we do not use a shooting method, but instead discretized the system, using orthogonal collocation and apply Newton’s method to solve for the discretized solution. Collocation can introduce hundreds of unknowns, but the use of a continuation method makes it practical to solve the system.

- Finally, we add an unfolding term to the equations to allow us to parameterize the family of periodic orbits implied by the existence of the Jacobi integral...
The Cylinder Theorem and Continuation of Periodic Solutions

**Cylinder Theorem**: An elementary periodic orbit of a system with an integral $I$ lies in a smooth cylinder of periodic solutions parameterized by $I$.

- The CR3BP is a conservative system, so each periodic solution lies in a branch without a parameter. As a result, the standard continuation method in AUTO2000 does not apply directly.

- To circumvent this problem we introduce an unfolding term to the equations to allow us to parameterize the family of periodic orbits implied by the Cylinder Theorem. This adds an unfolding parameter $\lambda$ to the system that allows the pseudo-arclength continuation method to be applied. A periodic solution can only occur if $\lambda = 0$. 
The Equations for the CR3BP

The equations we solve have an unfolding parameter \( \lambda \) and unfolding term.

\[
\begin{align*}
x' &= T v_x + \lambda \frac{\partial C}{\partial x}, \\
y' &= T v_y + \lambda \frac{\partial C}{\partial y}, \\
z' &= T v_z + \lambda \frac{\partial C}{\partial z}, \\
v'_x &= T [2v_y + x - (1 - \mu)(x + \mu)r_1^{-3} - \mu(x - 1 + \mu)r_2^{-3} + \lambda \frac{\partial C}{\partial v_x}], \\
v'_y &= T [-2v_x + y - (1 - \mu)yr_1^{-3} - \mu yr_2^{-3} + \lambda \frac{\partial C}{\partial v_y}], \\
v'_z &= T [-(1 - \mu)zr_1^{-3} - \mu zr_2^{-3} + \lambda \frac{\partial C}{\partial v_z}],
\end{align*}
\]

where \( T \) is the period of the periodic solution, and \( C \) is the Jacobi integral.
Periodic Solutions Emanating from $L_1$

We focus on the periodic orbits that emanate from the libration point $L_1$. There are two families of periodic solutions near $L_1$:

- The planar Lyapunov family
- The “Vertical” family arising from the vertical periodic solutions to the linearized dynamics.

We continue these families, and detect branches of periodic solutions that bifurcate from these families. In general we have only computed transcritical and pitchfork bifurcations, and not period-doubling, torus, and subharmonic bifurcations.
Planar Lyapunov Orbits from $L_1$

The initial solution close to the $L_1$ is found analytically and the rest of the solutions are computed using continuation. We stop near collision, where the system becomes singular.

Two families of orbits bifurcate from the Lyapunovs: the Halos at $B(L, H)$ and the “Axials” at $B(L, A)$. 
Northern Halo Orbits from $L_1$

The Northern Halo orbits for $L_1$. These orbits have been studied extensively by Farquhar, Howell, Simo, and others.

We do not pursue here the two families that branch from bifurcations $B(H, 1)$ (which leads to the $L_4$ & $L_5$ Verticals) and $B(H, 2)$. 
Axial Orbits Connecting Verticals and Lyapunovs

A family of orbits connects the second bifurcation point $B(L, A)$ on the Lyapunovs to the first bifurcation point $B(V, A)$ on the Verticals. These “Axial” orbits are axially symmetric about the $x$-axis. Similar Axial orbits are mentioned in Zagouras et al., 1979.
Bifurcation Diagram for $L_1$

To organize the data from our computations we use a bifurcation diagram. The cubes are the Lagrange points, the spheres are simple bifurcation points, the gray plane signifies planar orbits, and the colored curves represent families of periodic orbits.
Vertical Orbits from $L_1$

We follow two bifurcations along the Vertical family:

- $B(V, A)$ (Yellow) gives rise to the Axial orbits
- $B(V, BF)$ (Orange) gives rise to the “Backflip” orbits

These “Doubly Symmetric” orbits have been known at least since Bray and Goudas, 1966.
What is a Backflip?
Backflip Orbits arising from $L_1$

- The ”Backflip” orbits are named after the Backflip maneuvers, described by Uphoff in 1989 and used in the Wind mission about ten years later.

- The Backflip maneuver is as an extension of the Double Lunar Swingby (Farquhar & Dunham, 1981):
  - The first swingby sends a spacecraft from Earth-Moon orbit plane onto an arc beyond that plane, then back for a second encounter with the Moon.
  - The second swingby returns the spacecraft trajectory into the Earth-Moon orbit plane.

- Backflip orbits generalize the Backflip maneuver.
  - Each Backflip orbit consists of a Northern arc and a Southern arc in the rotating frame.
  - The two arcs are connected by the lunar swingbys.
  - One of the arcs may be nearly planar.
Example of a Backflip Orbit

Backflip orbit with 4:3 resonance, \( i.e. \), the period is \( \frac{4}{3} \) the Earth-Moon orbit period. Therefore the orbit is also periodic in the inertial frame.
Class 1 Backflip Orbits (Phases 1 and 2)

The family of Backflip orbits can be divided into five phases, based on variations in the $z$-amplitudes of the Northern (N) and Southern (S) arcs.

**Phase 1:**
Begins at bifurcation $B(V,BF)$. 
Northern arc rises to max $z$-amplitude at 1N, 
Southern arc moves outward to 1S. 
Retrograde orbits.

**Phase 2:**
Begins at orbit with arcs 1N and 1S. 
Northern arc drops to local min $z$-amplitude at 2N, 
Southern arc moves downward to 2S. 
Direct orbits.
Backflip - Junction of Phases 1 and 2

This orbit has a 3:2 resonance, and nearly lies in a vertical plane. The semicircular arc and each of the nearly linear segments takes 1/2 Earth-Moon orbit period to traverse one-way.

Rotating Frame   Earth Centered Inertial
Class 1 Backflip Orbits (Phases 3 and 4)

**Phase 3:**
Begins at bifurcation with arcs 2N and 2S.
Northern arc rises to $\max z$-amplitude at 3N,
Southern arc drops to $\max z$-amplitude at 3S.

**Phase 4:**
Begins at orbit with arcs 3N and 3S.
Northern arc drops to near $z = 0$ at 4N,
Southern arc forms a second loop and rises to 4S.
Backflip - Junction of Phases 3 and 4

Here both the Northern and Southern arcs are significantly out of plane. The period is 1.992 Earth-Moon orbit periods, so the orbit nearly closes in the inertial frame.
Class 1 Backflip Orbits (Phase 5)

Phase 5:
Begins at orbit with arcs 4N and 4S.
Ends in collision with Moon’s surface at 5N and 5S.
Northern arc moves outward to 5N.
Southern arc drops to 5S.
Backflip - Near Collision with the Moon

This is orbit passes within 0.6 km of the Moon. Thereafter the family of backflips orbits collides with the Moon.

Rotating Frame

Earth Centered Inertial
Bifurcation Diagram for $L_1$

Like the Halos, the Backflip family can be divided into two Classes related by reflection across the $z = 0$ plane. We call the orbits shown above the Class 1 Backflips, labelled $BF_1$ in the bifurcation diagram. As the diagram shows, the Backflip Classes end in collision with the Moon.
What Causes the Backflip Phase Transitions?

• The answer is unclear, but the transitions correlate to changes in the close approach distance to the Moon and the Jacobi integral.
  – The transition between phases 1 and 2 corresponds to a local minimum in the close approach distance to the Moon, when plotted as a function of the period. Orbits change from retrograde to direct that this junction.
  – The transition between phases 2 and 3 corresponds to the maximum in the Jacobi integral $C$.
  – The transition between phases 3 and 4 corresponds to an inflection point in the Jacobi integral.
  – The transition between phases 4 and 5 corresponds to a local minimum in the Jacobi integral.

• It would be interesting to investigate the geometry of close approach for the Backflips, looking at the B-plane parameters.
What Causes the Backflip Phase Transitions? (2)
Mission Applications

• Halo orbits allow spacecraft to remain near the line between the primaries. Valuable for studying the Sun-Earth interaction, for a communications relay, or for a gateway onto the Interplanetary Superhighway.

• Axial orbits might be used in missions similar to the Halos.

• Planar and Backflip orbits allow a spacecraft to explore geospace near the ecliptic.

• Vertical, Axial and Backflip orbits allow a spacecraft to explore geospace beyond the ecliptic plane.

• Backflip orbits can be used to explore both in and out of the ecliptic.
Conclusions and Future Work

- The AUTO2000 continuation procedure is robust and can easily be automated using the Python-based command language. In fact, the whole calculation shown in the bifurcation diagram can be done in batch mode.
- We would be very interested in taking the data we compute for the CR3BP and putting it into a high-fidelity model to see how the Backflip orbits are perturb.
- AUTO2000 can be used to compute Floquet multipliers and eigenvectors, which could be used to investigate stability and calculate invariant manifolds for these orbits.
- We are planning to perform continuation in $\mu$ to follow the various bifurcation points and gain an understanding of the form of the bifurcation diagram for all $\mu$ values.
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The End
But wait, there’s more!

The branch from bifurcation point $B(H,1)$ connects the Halos with the branch of Verticals from $L_4$ ($V_4$) and $L_5$, and ultimately connects with the branch of Verticals from $L_3$. 
And even more!
Tracking the Sun

The period of this Lyapunov orbit is such that the line of apsides of the ellipse rotates in the ECI frame at the same rate that the Earth rotates around the Sun. A spacecraft in this orbit could monitor the Earth’s geomagnetic tail.

Unfortunately, at closest approach the orbit passes 68 km beneath the Moon’s surface.
The size of dimension $n$ can vary greatly, depending on the kind of problem:

- $10^1$ to $10^2$ for intrinsically discrete, algebraic problems
- $10^2$ to $10^3$ for discretizations of ordinary differential equation boundary value problems
- $10^3$ to $10^k$ for discretizations of partial differential equation boundary value problems