LIBRATION POINT ORBITS AND APPLiCATIONS

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Dynamical Substitutes of the Libration Points for Simplified Solar System Models

Gerard Gómez
Departament de Matemàtica Aplicada i Anàlisi. Universitat de Barcelona

Josep J. Masdemont & José María Mondelo
Departament de Matemàtica Aplicada I. Universitat Politècnica de Catalunya

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Objectives of the work

- Provide methodology to generate simplified models suitable for the analysis of the motion of a s/c in the Solar System.

- The generated models form a sequence with the following properties:
  1. The sequence has increasing formal and dynamical complexity.
  2. The models are "between" the RTBP and the Newtonian model of motion in the Solar System.
  3. All the models are a perturbation (eventually large) of the RTBP.

- Computation of the dynamical substitutes of the collinear libration points for some models.
Introduction

Most of the well known restricted problems take as starting point the RTBP. The Hamilton function of this system is in sinodical coordinates

\[ H = \frac{1}{2} (p_x^2 + p_y^2 + p_z^2) + y p_x - x p_y - \mu \frac{1 - \mu}{((x-\mu)^2 + y^2 + z^2)^{1/2}} - \frac{\mu}{((x-\mu + 1)^2 + y^2 + z^2)^{1/2}}. \]

In order to get closer to more realistic situations, or simplifications useful for the analysis of the motion around \( m_2 \), this model is modified in different ways:

1. Hill’s model. Its obtained setting the origin at \( m_2 \), rescaling coordinates by a factor \( \mu^{1/3} \) and keeping only the dominant terms of the expanded Hamiltonian in powers of \( \mu^{1/3} \). The Hamiltonian function is

\[ H = \frac{1}{2} (p_X^2 + p_Y^2 + p_Z^2) + Y p_X - X p_Y - \frac{1}{(X^2 + Y^2 + Z^2)^{1/2}} - X^2 + \frac{1}{2} (Y^2 + Z^2). \]

Hill’s model has a remarkable family of \( 2\pi m \)–periodic solutions, known as the Variation Orbit Family.
2. Assuming that the primaries move in an elliptic orbit instead of a circular one. This is what is known as the elliptic RTBP. Its main difference with the RTBP is that it is non-autonomous and in fact is a time-periodic perturbation of it.

3. Restricted Hill four body problem (Scheeres, 1998). This is a time periodic model that contains two parameters: the mass ratio $\mu$ of the RTBP and the period parameter $m$ of the Hill Variation Orbit. As $m \to 0$, the RTBP is recovered and the classical Hill model is recovered as $\mu \to 0$.

4. The Bicircular Restricted Problem. It can be seen as a periodic perturbation of the the RTBP in which one primary has been splitted in two that move around their common center of mass. This model can be suitable to take into account the gravitational effect of the Sun in the Earth–Moon RTBP. The Hamiltonian is

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + yp_x - xp_y - \frac{1 - \mu}{((x - \mu)^2 + y^2 + z^2)^{1/2}} \frac{\mu}{m_S} \frac{1}{((x - \mu + 1)^2 + y^2 + z^2)^{1/2}}$$
$$- \frac{m_S}{a_S} (y \sin \theta - x \cos \theta),$$

with $\theta = w_S t + \theta_0$, $w_S =$ mean angular velocity of the Sun, $m_S =$ Sun mass, $a_S =$ distance from the Earth–Moon barycenter to the Sun.

\[
H = \frac{1}{2} \alpha_1 (p_x^2 + p_y^2 + p_z^2) + \alpha_3 (yp_x - xp_y) + \alpha_2 (xp_x + yp_y + zp_z) + \alpha_4 x + \alpha_5 y - \alpha_6 \left( \frac{1 - \mu}{((x - \mu)^2 + y^2 + z^2)^{1/2}} + \frac{\mu}{((x - \mu + 1)^2 + y^2 + z^2)^{1/2}} + \frac{m_S}{((x - \alpha_7)^2 + (y - \alpha_8)^2 + z^2)^{1/2}} \right),
\]

where the \( \alpha_i \) are time periodic functions, with the same basic frequency as the Bicircular Problem.
Strategy for the construction of the models

Main steps (following Gómez, Llibre, Martínez, Simó, 1985):

1. Using suitable coordinate changes, write the Newtonian equations of motion in the Solar System as a perturbation of the RTBP.

2. Perform a “refined Fourier analysis” (Mondelo, 2001; Gómez, Mondelo, Simó, 2001) of the time-dependent functions that appear in the new equations.

3. Adequate selection of a frequency basis.

4. Approximation of the time-dependent functions by their trigonometric approximations with an increasing number of frequencies. Construction of the models.
Coordinate changes

Staring equations: Newton’s equation for the motion of an infinitesimal body in the Solar System

\[
\mathbf{R}'' = G \sum_{i \in S} m_i \frac{\mathbf{R}_i - \mathbf{R}}{||\mathbf{R} - \mathbf{R}_i||^3},
\]

\( G \) = gravitational constant,
\( \mathbf{R} \) = position of the infinitesimal body,
\( \mathbf{R}_i \) = position of the Solar System body \( i \),
\( m_i \) = mass of the Solar System body \( i \).

1. Select two bodies as primaries \( I, J \in S \) with \( m_I > m_J \). In this way, \( \mu = m_J/(m_I + m_J) \), and \( 1 - \mu = m_I/(m_I + m_J) \).

2. Introduce a synodic reference frame. The transformation from synodical coordinates, \( \mathbf{r} = (x, y, z)^T \), to sidereal ones, \( \mathbf{R} \), is defined by

\[
\mathbf{R} = \mathbf{B} + k \mathbf{C} \mathbf{r},
\]

where

- The translation, \( \mathbf{B} \), that sets the barycenter of the primaries at the origin, is given by

\[
\mathbf{B} = \frac{m_I \mathbf{R}_I + m_J \mathbf{R}_J}{m_I + m_J}.
\]
– The orthogonal matrix $C = (e_1, e_2, e_3)$, sets the primaries on the $x$–axis and turns the instantaneous plane of motion of the primaries into the $xy$ plane. The columns of $C$ are

$$e_1 = \frac{R_{J'I}}{\|R_{J'I}\|}, \quad e_3 = \frac{R_{J'I} \times R'_{J'I}}{\|R_{J'I} \times R'_{J'I}\|}, \quad e_2 = e_3 \times e_1,$$

being $R_{J'I} = R_J - R_I$.

– $k = \|R_{J'I}\|$ is a scaling factor which makes the distance between the primaries to be constant and equal to 1.

3. Introduce adimensional time units (as in the RTBP). If $t^*$ is some dynamical time and $n$ is the mean motion of $J$ with respect to $I$ (computed, for instance, using the JPL ephemeris files), then

$$t = n(t^* - t_{0^*}),$$

where $t_{0^*}$ is a fixed epoch.

**Remark:** The change of variables is non–autonomous!!
The Solar System as RTBP+pert.

With the above transformations, the second–order differential equations of motion of the Solar System are

\[
\begin{align*}
\ddot{x} &= b_1 + b_4 \dot{x} + b_5 \dot{y} + b_7 x + b_8 y + b_9 z + b_{13} \frac{\partial \Omega}{\partial x}, \\
\ddot{y} &= b_2 - b_5 \dot{x} + b_4 \dot{y} + b_6 \dot{z} - b_8 x + b_{10} y + b_{11} z + b_{13} \frac{\partial \Omega}{\partial y}, \\
\ddot{z} &= b_3 - b_6 \dot{y} + b_4 \dot{z} + b_9 x - b_{11} y + b_{12} z + b_{13} \frac{\partial \Omega}{\partial z},
\end{align*}
\]

being

\[
\Omega = \frac{1 - \mu}{\sqrt{(x - \mu)^2 + y^2 + z^2}} + \frac{\mu}{\sqrt{(x - \mu + 1)^2 + y^2 + z^2}} + \sum_{i \in S'} \frac{\mu_i}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}}
\]

where \( \mu_i = m_i/(m_I + m_J) \) and the \( b_i \) time–dependent functions are defined as

\[
\begin{align*}
b_1 &= -\frac{1}{k} \langle \dot{\mathbf{B}}, \mathbf{e}_1 \rangle, \\
b_2 &= -\frac{1}{k} \langle \dot{\mathbf{B}}, \mathbf{e}_2 \rangle, \\
b_3 &= -\frac{1}{k} \langle \dot{\mathbf{B}}, \mathbf{e}_3 \rangle, \\
b_4 &= -\frac{2k}{k}, \\
b_5 &= 2 \langle \dot{\mathbf{e}}_1, \mathbf{e}_2 \rangle, \\
b_6 &= 2 \langle \dot{\mathbf{e}}_2, \mathbf{e}_3 \rangle, \\
b_7 &= \langle \dot{\mathbf{e}}_1, \dot{\mathbf{e}}_1 \rangle - \frac{\ddot{k}}{k}, \\
b_8 &= \ldots
\end{align*}
\]

Remark: Setting \( b_i = 0 \) for \( i \neq 5, 7, 10, 13, b_5 = 2, b_7 = b_{10} = b_{13} = 1 \) and skipping the sum over \( S' \) the RTBP is recovered.
Fourier analysis of the perturbation

Using the algorithm described in (Mondelo, 2001; Gómez et al., 2001) we perform Fourier analysis of the \( \{b_i\} \) and \( x_i \) functions, for a given set of primaries.

The most relevant parameters to be specified are:

- The size, \( T \), of the time (sampling) interval,
- The number, \( N \), of equally spaced sampling points chosen in the interval.

These parameters define the Nyquist critical frequency, \( \omega_c = N/(2T) \), that fixes the window within we will find all the frequencies (true or aliased) of our time series.

- The parameter \( N \) ranges over powers of two.
- The length, \( T \), of the time–interval has also been chosen to range over a geometric progression.
- The smallest time–interval length, \( T_{\text{min}} \), has been taken of 95 years (34698.75 Julian days) and the greatest time–interval length, \( T_{\text{max}} \), has been taken of 364938 Julian days (999.15 years).
- The maximum number of samples \( N_{\text{max}} \) has been chosen to be \( 2^{20} \), in order to allow for “comfortable” runs. For each value of \( T \), the minimum number of samples has been chosen such that \( \frac{N}{2T} \geq 1.5 \), in order to make the maximum detectable frequency to be at least 1.5.
### Anti–aliasing strategies

Since it is based on the Discrete Fourier Transform (DFT), the refined Fourier procedure is subject to aliasing (that is, any frequency outside $[0, N/(2T)]$ is seen as an “aliased frequency” inside this interval).

The anti–aliasing strategy used consists in computing the number of rightmost consecutive harmonics of the residual DFT that have modulus less than a fraction of the maximum modulus of the residual DFT. Then, we divide this number by $N/2$, the total number of harmonics, and this defines the parameter $\alpha_2$.

For instance, a value of 0.2 for $\alpha_2$ means that there are no frequencies greater than $0.8 \cdot \omega_{\text{max}} = 0.8 \cdot (N/2T)$, with amplitude greater than $1/25$ times the modulus of the residual DFT, so we do not expect aliasing in the corresponding Fourier analysis.

![Modulus of the residual DFT some of the Fourier analysis of the $b_1$ function in the Earth–Moon case.](image)
## Fourier Analysis results

**(Earth–Moon case)**

<table>
<thead>
<tr>
<th>function</th>
<th>$T$ (days)</th>
<th>$T$ (J–rev.)</th>
<th>$N$</th>
<th>$d_{\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_1$</td>
<td>55551.4</td>
<td>2033.24</td>
<td>65536</td>
<td>5.63E–04</td>
</tr>
<tr>
<td>$b_2$</td>
<td>55551.4</td>
<td>2033.24</td>
<td>65536</td>
<td>5.49E–04</td>
</tr>
<tr>
<td>$b_3$</td>
<td>55551.4</td>
<td>2033.24</td>
<td>32768</td>
<td>5.58E–05</td>
</tr>
<tr>
<td>$b_4$</td>
<td>55551.4</td>
<td>2033.24</td>
<td>65536</td>
<td>5.01E–05</td>
</tr>
<tr>
<td>$b_5$</td>
<td>43904.0</td>
<td>1606.94</td>
<td>32768</td>
<td>9.16E–05</td>
</tr>
<tr>
<td>$b_6$</td>
<td>70288.7</td>
<td>2572.64</td>
<td>32768</td>
<td>1.13E–06</td>
</tr>
<tr>
<td>$b_7$</td>
<td>55551.4</td>
<td>2033.24</td>
<td>65536</td>
<td>7.81E–05</td>
</tr>
<tr>
<td>$b_8$</td>
<td>55551.4</td>
<td>2033.24</td>
<td>524288</td>
<td>5.94E–06</td>
</tr>
<tr>
<td>$b_9$</td>
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<td>2572.64</td>
<td>65536</td>
<td>5.69E–05</td>
</tr>
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<td>$b_{10}$</td>
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<td>2033.24</td>
<td>65536</td>
<td>7.83E–05</td>
</tr>
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<td>$b_{11}$</td>
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<td>2572.64</td>
<td>65536</td>
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</tr>
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<td>1606.94</td>
<td>32768</td>
<td>3.29E–05</td>
</tr>
<tr>
<td>$b_{13}$</td>
<td>55551.4</td>
<td>2033.24</td>
<td>65536</td>
<td>7.99E–05</td>
</tr>
</tbody>
</table>

Values of the parameters for the best Fourier analyses of the $b_i$ functions for the Earth–Moon case.

In the above table

$$d_{\text{max}} = \max_{l=0,\ldots,N-1} |b_i(t_l) - Q_{b_i}(t_l)|,$$

where $t_l = l(T/N)$, $l = 0,\ldots,N-1$ are the sampling epochs and $Q_{b_i}(t)$ is the trigonometric approximation of $b_i(t)$. 
Choosing basic frequencies

In order to give a more physical meaning to the results obtained from the Fourier analysis, we will write the computed frequencies as linear combinations, with integer coefficients, of basic ones.

In principle, the basic frequencies will be extracted from the list of frequencies computed in the Fourier analysis and using the following procedure:

Assume that $\omega_1, \ldots, \omega_n$ is a set of basic frequencies and that a frequency $f$ can be written as

$$f = k_1 \omega_1 + \ldots + k_n \omega_n, \quad k_1, \ldots, k_n \in \mathbb{Z}$$

then we say that $f$ is a linear combination of $\omega_1, \ldots, \omega_n$ of order $k = |k_1| + \ldots + |k_n|$.

We say that $f$ is a linear combination of $\omega_1, \ldots, \omega_n$ of order $k$ within tolerance $\varepsilon > 0$ if, for some $k_1, \ldots, k_n$ such that $k = |k_1| + \ldots + |k_n|$, we have

$$|f - (k_1 \omega_1 + k_2 \omega_2 + \ldots + k_n \omega_n)| < \varepsilon.$$
PROCEDURE:

1. Choose a maximum order of the linear combinations to be found.

2. Choose a tolerance for the adjustment of frequencies as linear combination of basic ones.

3. For each frequency, try out all the linear combinations of the current set of basic frequencies up the chosen maximum order.

4. If any of the linear combinations fulfills the tolerance requirements, add the current frequency to the set of basic ones.

This procedure may add extra basic frequencies (and end up with a rationally dependent set), for instance, if the current frequency is an integer divisor of one of the basic frequencies. To avoid this, we will try to adjust zero as linear combination of the current frequency and the basic ones. If we succeed and the current frequency gets a coefficient different from $\pm 1$, it may be necessary to divide some basic frequencies by this coefficient.
In some cases it can be convenient to introduce a fixed set of basic frequencies obtained by other means, for instance from an analytical lunar theory, and then write all the computed frequencies as linear combinations of the ones in this fixed set.

For the Earth–Moon system and according to Escobal, the fundamental parameters can be expressed in terms of five basic frequencies. In terms of cycles per lunar revolution, their numerical values are

- \( \omega_1 = 1.0 \) (mean longitude of the Moon).
- \( \omega_2 = 0.925195997455093 \) (mean elongation of the Moon from the Sun).
- \( \omega_3 = 8.45477852931292 \times 10^{-3} \) (mean longitude of the lunar perigee).
- \( \omega_4 = 4.01883841204748 \times 10^{-3} \) (longitude of the mean ascending node of the lunar orbit on the ecliptic).
- \( \omega_5 = 3.57408131981537 \times 10^{-6} \) (Sun’s mean longitude of perigee).

Now we need to obtain a new basis that allows to generate a sequence of models that successively improve the RTBP to approach the real Solar System.
We take $\nu_1 = \omega_2$ as the first frequency of the new basis, because it is the main frequency of $b_1, b_2, x_S$ and $y_S$, and, in this way, it can be considered the main “planar frequency”.

The main frequencies of most of the the remaining functions can be expressed as linear combinations of $\omega_2$ and $\omega_1 - \omega_3$. Thus, we will take $\nu_2 = \omega_1 - \omega_3$.

The remaining $\nu_i$ have been taken as:

- $\nu_3 = \omega_1 - \omega_2 + \omega_4$, which is the main frequency of $b_3$,
- $\nu_4 = \omega_1 - \omega_5$, which is the first frequency of $x_S$ which cannot be expressed in terms of $\nu_1, \nu_2$, and
- $\nu_5 = \omega_5 - \omega_2$, which is the first frequency of $b_3$ that cannot be expressed in terms of $\nu_1, \nu_2, \nu_3, \nu_4$

In this way, we have

$$
\begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3 \\
\nu_4 \\
\nu_5
\end{pmatrix} =
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & -1 \\
0 & -1 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\omega_1 \\
\omega_2 \\
\omega_3 \\
\omega_4 \\
\omega_5
\end{pmatrix}.
$$

Since the matrix in the above transformation is unimodular, $\{\nu_i\}_{i=1}^{5}$ is a valid basic set of frequencies.
Construction of the models

Once we have the definitive basis \( \{ \nu_i \}_{i=1,\ldots,5} \), we define our simplified Solar System models, which will be called SSSM\(_i\), as

\[
\begin{align*}
\dot{x} &= b_1^{(i)} + b_4^{(i)} \dot{x} + b_5^{(i)} \dot{y} + b_7^{(i)} x + b_8^{(i)} y + b_9^{(i)} z + b_{13}^{(i)} \frac{\partial \Omega^{(i)}}{\partial x}, \\
\dot{y} &= b_2^{(i)} - b_5^{(i)} \dot{x} + b_4^{(i)} \dot{y} + b_6^{(i)} \dot{z} - b_8^{(i)} x + b_{10}^{(i)} y + b_{11}^{(i)} z + b_{13}^{(i)} \frac{\partial \Omega^{(i)}}{\partial y}, \\
\dot{z} &= b_3^{(i)} - b_6^{(i)} \dot{y} + b_4^{(i)} \dot{z} + b_9^{(i)} x - b_{11}^{(i)} y + b_{12}^{(i)} z + b_{13}^{(i)} \frac{\partial \Omega^{(i)}}{\partial z},
\end{align*}
\]

being

\[
\Omega^{(i)} = \frac{1 - \mu_{E,M}}{\sqrt{(x - \mu_{E,M})^2 + y^2 + z^2}} + \frac{\mu_{E,M}}{\sqrt{(x - \mu_{E,M} + 1)^2 + y^2 + z^2}} + \frac{\mu_{E,M,S}}{\sqrt{(x - x_S^{(i)})^2 + (y - y_S^{(i)})^2 + (z - z_S^{(i)})^2}},
\]

where \( b_j^{(i)} \), \( x_S^{(i)} \), \( y_S^{(i)} \), \( z_S^{(i)} \) stand for the computed Fourier expansions of \( b_j \), \( x_S \), \( y_S \), \( z_S \) but keeping only the terms with frequencies that are linear combinations of the first \( i \) basic frequencies.
Dynamical substitutes of $L_{1,2,3}$ in $SSSM_1$

The dynamical substitutes of $L_{1,2,3}$ in $SSSM_1$ are periodic orbits. They are obtained from the collinear equilibria of the RTBP by continuation w.r.t. $\varepsilon$ through the family of models $(1 - \varepsilon)RTBP + \varepsilon SSSM_1$.

The continuation of $L_{1,3}$ is straightforward. The $L_2$ case is somewhat more intricate.
Numerical computation of two–dimensional invariant tori

The dynamical substitutes of $L_{1,2,3}$ in $SSM_2$ are two–dimensional invariant tori. They are computed as follows (Castellà & Jorba, 2000, Gómez & Mondelo, 2001): the $SSM_2$ model can be written as

$$\dot{x} = f(x, \nu t)$$

where $\nu = (\nu_1, \nu_2)$ and $f$ is $2\pi$–periodic in $\nu t$. We do not actually compute a 2D invariant torus but an invariant curve inside it. For that, we solve numerically for $\phi$ the equation

$$\phi(\xi + \rho) = F(\phi(\xi), \xi), \quad \forall \xi \in \mathbb{T}^1,$$

being $\rho = \frac{2\pi}{\nu_1}$ and $F(x, \xi) = \phi^{(0,\xi)}_\rho(x)$, where $\phi^{\theta}_t(x)$ is the flow from time 0 to time $t$ of

$$\dot{x} = f(x, \theta + \nu t)$$

(which is not $SSM_2$ if $\theta \neq 0$). The geometrical torus is then

$$\{ \phi^{(0,\theta_2)}_{\theta_1,\nu_1}(\phi(\theta_2)) \}_{(\theta_1,\theta_2) \in \mathbb{T}^e}. $$
Dynamical substitutes of $L_{1,2,3}$ in SSSM$_2$

They can be computed from the corresponding substitutes of SSSM$_1$ by continuation with respect to $\varepsilon$ through the family of models

$$(1 - \varepsilon)\text{SSSM}_1 + \varepsilon\text{SSSM}_2$$

As an example, we display the continuation corresponding to the $L_1$ case.
Conclusions and outlook

• We have provided methodology for the development of sequences of models, with increasing complexity, that “fill the gap” between the RTBP around a given pair of primaries and the real Solar System.

• This methodology has been applied to the development of a sequence of 5 models for the Earth–Moon system.

• For the first two, the dynamical substitutes of the collinear libration points have been computed.

• In the future, a systematic study of the neighborhood of the collinear libration points will be done for these models. This will require the development of methodology for the numerical computation of invariant tori of more than 2 dimensions.